Combinatorics

7.1 INTRODUCTION

In many discrete problems, we are confronted with problem of counting. Combinatorics is that branch of discrete mathematics concerned with counting problems. Techniques for counting are important in Mathematics and Computer Science especially in probability theory and in the analysis of algorithms. In this chapter, we will discuss basics of counting and their applications.

7.2 THE FUNDAMENTAL COUNTING PRINCIPLES

Two elementry principles act as building bolcks for all counting problems.

7.2.1 Disjunctive (or) Sum Rule

If an event can occur in m ways and another event can occur in n ways, and if these two events cannot occur simultaneously, then one of the two events can occur in $m + n$ ways. In general, if E_i ($i = 1, 2, 3, ..., k$) are k events such that no two of them can occur at the same time, and if E_i can occur in n_i ways, then one of the k events can occur in $n_1 + n_2 + n_3 + ...+ n_k$ ways.

Solved Examples

Example 1. If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student as class representative.

Solution. Using sum rule, there are $14 + 12 = 26$ ways of selecting one student (either a boy or a girl) as class representative.

Example 2. If a student is getting admission in 4 different Engineering Colleges and 5 Medical Colleges, find the number of ways of choosing one of the above colleges.

Solution. Using sum rule, there are $4 + 5 = 9$ ways of choosing one of the colleges.

7.2.2 Sequential (or) Product Rule

If an event can occur in m ways and a second event in n ways, and if the number of ways the second event occurs, does not depend upon the occurrence of the first event, then the two events can occur simultaneously in mn ways. In general if E_i ($i = 1, 2, ..., k$) are k events and if E_1 can occur in n_1 ways, E_2 can occur in n_2 ways (no matter how E_1 occurs), E_3 can occur in n_3 ways (no matter how E_1 and E_2 occurs),.....,E_k occurs in n_k ways (no matter how $E_1, E_2, ..., E_{k-1}$ events occur), then the k events can occur simultaneously in $n_1 \times n_2 \times n_3 \times ....\times n_k$ ways.

Example 3. Three persons enter into car, where there are 5 seats. In how many ways can they take up their seats?

Solution. The first person has a choice of 5 seats and can sit in any one of those 5 seats. So there are 5 ways of occupying the first seat. The second person has a choice of 4 seats. Similarly, the third person has a choice of 3 seats. Hence, the required number of ways in which all the three persons can seat is $5 \times 4 \times 3 = 60$.

Example 4: In how many ways can three different coins be placed in two different purses?

[JNTU(K) Oct 2017 (Set No. 3)]

Solution: The required number $= 3 \times 2 = 6$, using the Product Rule

Example 5. There are four roads from city X to Y and five roads from city Y to Z, find

(*i*) how many ways is it possible to travel from city X to city Z via city Y.

(*ii*) how different round trip routes are there from city X to Y to Z to Y and back to X.

Solution. (*i*) In going from city X to Y, any of the 4 roads may be taken. In going from city Y to Z, any of the 5 roads may be taken. So by the product rule, there are $5.4 = 20$ ways to travel from city X to Z via city Y.

(*ii*) A round trip journey can be performed in the following four ways:

1. From city X to Y
2. From city Y to Z
3. From city Z to Y
4. From city Y to X

1. Can be performed in 4 ways, 5 ways to perform 2, 5 ways to perform 3 and 4 ways to perform 4. By product rule, there are $4.5.5.4 = 400$ round trip routes.

Example 6: There are four bus lines between A and B and five bus lines between B and C. In what many ways can a man travel round trip by bus from A to C via B if he does not want to use a bus line more than once? *[JNTU(K) Oct 2018(Set No. 1)]*

Solution: The man can travel from A to B in four ways and from B to C in five ways, but only four ways from C to B and only in three ways from B to A if he does not use a route more than once.

Hence, required number of ways he can make the round trip from A to C via $B = 4 \times 5 \times 4 \times 3 = 240$.

Example 7. For a set of six true or false questions, find the number of ways of answering all questions.

Solution. The number of ways of answering the first question is 2. The second question can also be answered in 2 ways and similarly for other 4 questions. Hence the total number of ways of answering all the questions is $2^6 = 64$.

Many counting problems can be solved using just the sum rule or just the product rule. However, many counting problems can be solved using both of these rules.

Example 8. In how many a ways can one select two books from different subjects from among six distinct computer science books, three distinct mathematics books, and two distinct chemistry books?

Solution. Using product rule one can select two books from different subjects as follows:

(*i*) one from computer science and one from mathematics in 6.3 = 18 ways.

(*ii*) one from computer science and one from chemistry in 6.2 = 12 ways.

(*iii*) one from mathematics and one from chemistry in 3.2 = 6 ways.

Since these sets of selections are pairwise disjoint, one can use the sum rule to get the required number of ways which is $18 + 12 + 6 = 36$.

Example 9. Find the number of three digit even numbers with no repeated digits

[*JNTU(K)Oct.* 2017(*Set No.* 2)]

Solution: Let the required numbers of the form x, y, z where each of x, y, z represents a digit which are all different. Since the required number has to be even, the unit place *i.e.*, z has to be 0, 2, 4, 6, or 8. If z is 0, then x has 9 choices and if z is 2, 4, 6, or 8 (4 choices) then x has 8 choices. Remember that x cannot be zero. Therefore, z and x can be chosen in $(1 \times 9) + (4 \times 8) = 41$ ways. For each of these ways, y can be choosen 8 ways. Hence, the required number is $41 \times 8 = 328$

Example 10. How many different five digit numbers can be formed from the digits 0, 1, 2, 3 and 4? **[*JNTU(K) Oct. 2019 (Set No. 3), (R19) March 2021*]**

Solution: We know that 0 cannot be used for 10^4 place, So 4 digits can be used for the 10^4 place, 4 for 10^3 place, 3 for 10^2 place and 2 for 10th place.

$\therefore$ The desired number $= 4 \times 4 \times 3 \times 2 \times 1 = 96$

Example 11. How many different six digit numbers can be formed from the digits 0, 1, 2, 3, 4 and 5?

Solution: As explained in the previous example, the required no. of ways

$= 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$

Example 12. How many different outcomes are possible by tossing 10 similar coins?

[*JNTU(A) (R19) Aug. 2021* (Sup)]

Solution: When a coin is tossed the outcomes are heads and tails. There are two ways of arranging head and tail in a sequence.

$\therefore$ Required number $= 2^{10}$

7.3 FACTORIAL NOTATION

The product of n consecutive positive integers beginning with 1 is denoted by $n!$ or $\lfloor\underline{n}$ and is read as factorial n. Thus

$$n! = 1.\,2.\,3.\,\ldots\ldots\,(n-1)n = n(n-1)(n-2)\,\ldots\ldots\,3.\,2.\,1.$$

For example,
$$5! = 1.\,2.\,3.\,4.\,5 = 5.\,4.\,3.\,2.\,1 = 120$$
$$4! = 1.\,2.\,3.\,4 = 4.\,3.\,2.\,1 = 24$$

Clearly,
$$\begin{aligned} n! &= 1.\,2.\,3.\,\ldots..\,(n-2)(n-1)n \\ &= \{1.\,2.\,3.\,\ldots\ldots\,(n-2)(n-1)\}n \\ &= (n-1)!\,n = n.\,(n-1)!. \end{aligned}$$

Thus,
$$\begin{aligned} n! &= n(n-1)! = n.(n-1)(n-2)!. \\ &= n(n-1)(n-2)(n-3)! \text{ etc.} \end{aligned}$$

It r and n are positive integers and $r < n$, then

$\frac{n!}{r!} = n(n-1)(n-2) \ldots\ldots (r+1)$ and $\frac{n!}{(n-r)!} = n(n-1)(n-2) \ldots\ldots (n-r+1)$

For example, $\frac{5!}{2!} = 5.\,4.\,3 = 60$, $\frac{3!}{6!} = \frac{1}{6.\,5.\,4} = \frac{1}{120}$

Example 13. 36 cars are running between two places P and Q. In many ways can a person go from P to Q and return by a different car. [*JNTU (A) Nov. 2018*]

Solution: Since a person can go from P and Q in 36 ways and come back in 35 ways, the total number of ways is $36 \times 35 = 1260$

Example 14. Prove that $(2n)! = 2^n\, n!\, \{1.\,3.\,5.\, \ldots\ldots (2n-1)\}$

Solution.
$$\begin{aligned}(2n)! &= 2n\,(2n-1)(2n-2) \ldots\ldots 5.\,4.\,3.\,2.\,1 \\ &= [2n\,(2n-2) \ldots\ldots 4.\,2]\,[(2n-1)(2n-3) \ldots\ldots 5.\,3.\,1] \\ &= 2^n\,[n(n-1) \ldots\ldots 2.\,1]\,\{1.3.5.\, \ldots\ldots (2n-3)(2n-1)\} \\ &= 2^n\, n!\, \{1.\,3.\,5.\, \ldots\ldots (2n-3)(2n-1)\}\end{aligned}$$

Hence proved.

7.4 PERMUTATION

The different arrangements which can be made out of a given set of things, by taking some or all of them at a time are called their permutations.

The number of permutations of n different things taken $r(\leq n)$ at a time is denoted by $P(n, r)$ or ${}^nP_r$. For example,

The different arrangement of three letters a, b, c taking two at a time are ab, bc, ca, ba, cb, ac and thus the number of arrangements (or permutations) of 3 things taken 2 at a time is $P(3, 2) = 6$.

7.4.1 The value of $P(n, r)$ or ${}^nP_r$

To find the number of permutations of n different things taken $r(\leq n)$ at a time.

The number of permutations of n things taken r at a time is the same as the number of ways in which r blank places can be filled up with n given things.

The first place can be filled up in n ways since any one of the n different things can be put in it.

After having filled up the first place by any one of the n things, there are $(n-1)$ things left. Hence the second place can be filled up in $(n-1)$ ways. Now, corresponding to each way of filling up the first place, there are $(n-1)$ ways of filling up the second place, so the first two places can be filled up in $n(n-1)$ ways.

After having filled up the first two places in any one of the above ways, there are $(n-2)$ things left and so the third place can be filled up in $(n-2)$ ways. Now corresponding to each ways of filling up the first two places, there are $(n-2)$ ways of filling up the third place and so the first three places can be filled up in $n(n-1)(n-2)$ ways.

It may be observed that

(*i*) at every stage the number of factors is equal to the number of places filled up.

(ii) each factor is less than the proceeding factor by 1.

Place	1st	2nd	...	$(r-1)$th	rth
Number of ways	n	$n-1$	...	$n-(r-2)$	$n-(r-1)$

Hence the number of ways of filling up all the r places *i.e.*, the number of permutations of different things taken r at a time

$$= n(n-1)(n-2) \text{ ...... to } r \text{ factors}$$
$$= n(n-1)(n-2) \text{ ..... } [n-(r-1)]$$

Hence $\boxed{P(n, r) = n(n-1)(n-2) \text{ ...... } (n-r+1)}$

Now, $P(n, r) = \dfrac{n(n-1)(n-2)\text{.........}(n-r+1)(n-r)!}{(n-r)!}$

$\therefore \quad P(n, r) = \dfrac{n!}{(n-r)!}$ ... (1)

Note 1. Number of permutations of n different things taken all n at a time is

$P(n, n) = n(n-1)(n-2) \text{ ...... } 3.\ 2.\ 1 = n!$ ... (1)

That is, the number of different arrangements or permutations of n distinct objects taken all at a time is $n!$. This is simply called the number of permutations of n distinct objects.

Note 2. Meaning of 0!

Putting $r = n$ in (1), we obtain

$$P(n, n) = \frac{n!}{(n-n)!}$$

$\Rightarrow \quad n! = \dfrac{n!}{0!}$ *i.e.*, $0! = \dfrac{n!}{n!} = 1$

Note 3. $P(n, r) = n \times P(n-1, r-1)$.

$$n \times P(n-1, r-1) = n \times \frac{(n-1)!}{(n-1-r+1)!} = n \times \frac{(n-1)!}{(n-r)!} = \frac{n!}{(n-r)!} = P(n, r)$$

Example 15. If ${}^nP_2 = 72$, find the value of n.

Solution: Here ${}^nP_2 = 72 \Rightarrow \dfrac{n!}{(n-2)!} = 72$

$\therefore \quad \dfrac{n \times (n-1) \times (n-2)!}{(n-2)!} = 72 \Rightarrow n(n-1) = 72$

$n^2 - n - 72 = 0 \Rightarrow n^2 - 9n + 8n - 72 = 0$

$(n-9)(n+8) = 0 \therefore \ n = 9, -8$

But $n \neq -8$ (n cannot be negative). So $n = 9$.

Example 16. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, find the value of n.

Solution: Here ${}^{2n+1}P_{n-1} = \dfrac{(2n+1)!}{(n+2)!}$ and ${}^{2n-1}P_n = \dfrac{(2n-1)!}{(n-2)!}$

$\therefore$

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

or $\quad \dfrac{(2n+1)\times 2n\times(2n-1)!}{(n+2)\times(n+1)\times n\times(2n-1)!}=\dfrac{3}{5}\Rightarrow\dfrac{(2n+1)\times 2n}{(n+2)\times(n+1)\times n}=\dfrac{3}{5}$

or $\quad \dfrac{4n+2}{n^2+3n+2}=\dfrac{3}{5}\Rightarrow 20n+10=3n^2+9n+6$

or $\quad 3n^2-11n-4=0$

or $\quad (n-4)(3n+1)=0 \therefore n=4, -\dfrac{1}{3}$

But, $n\neq-\dfrac{1}{3} \therefore n=4$

Example 17. In how many ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only even positions? *[JNTU(K) Oct. 2018 (Set No. 3)]*

Solution: Number of consonants in the given word = 3

Number of even places = 3

$\therefore$ The desired number = $3_{P_3}\times 3! = 3!\times 3! = 6\times 6 = 36$

Example 18. In how many ways can the letters of the word 'MATRIX' be arranged so that the vowels occupy only the even positions?

Solution: Number of vowels in the given word = 2

Number of even places = 3

$\therefore$ The desired number = $3_{P_2}\times 4! = 6\times 24 = 144$

Example 19. In how many ways can be the letters of the word 'TRIANGLE' so that the relative positions of vowels and consonants remain unaltered?

Solution: Number of vowels in the given word = 3

Number of consonants in the given word = 5

$\therefore$ The desired number = $3!\times 5! = 6\times 120 = 720$

Example 20. In how many ways can the letters of the word 'DISCRETE' so that the vowels may come together?

Solution: No. of letters in the given word = 8

No. of vowels in the given word = 3

$\therefore$ The required number = $3!\times 5! = 6\times 120 = 720$

Example 21. How many different numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4 and 5, no digit being repeated.

Solution: The numbers lying between 100 and 1000 are of 3 digits. They will be formed by taking 3 digits from the given 5 digits. So the required number is equal to the number of permutations of 5 different things taken 3 at a time i.e. $P(5, 3) = \dfrac{5!}{2!} = 5.4.3 = 60$

Example 22. Three prizes are to be awarded among 10 candidates. In how many ways can the prizes be given, so that no candidate may get more than one prize?

Solution: Since no candidate get more than one prize, the required number is equal to the number of permutations of 10 different things taken 3 at a time *i.e.*,

$$P(10,3) = \frac{10!}{7!} = 10 \times 9 \times 8 = 720.$$

7.4.2 Restricted Permutations

1. The number of Permutations of n different objects taken r at a time in which k particular objects do not occur is $\mathbf{P(n-k, r)}$.

2. The number of permutations of n different objects taken r at a time in which k particular objects are always present is $\mathbf{P(n-k, r-k) \times P(r, k)}$.

Example 23. In how many of the permutations of 10 things taken 4 at a time will **(*a*)** two things always occur (*b*) never occur ?

Solution. (*a*) Keeping aside the two particular things which will always occur, the number of permutations of $10 - 2 = 8$ things taken 2 at a time $^8P_2$. Now 2 particular things can take up any one of the four places and so can be arranged in $^4P_2$ ways.

Hence the total numbers of permutation is $^8P_2 \times {}^4P_2 = 8 \times 7 \times 4 \times 3 = 672$

(*b*) Leaving aside the two particular things which will never occur, the number of permutations of 8 things taken 4 at a time $= {}^8P_4 = 8 \times 7 \times 6 \times 5 = 1680$.

3. When certain things not occurring together

Example 24. Prove that the number of ways in which n books can be arranged on a shelf so that two particular books are never together is $(n-2) \times (n-1)!$.

Solution. Treating the two particular books as one book, there are $(n-1)$ books which can be arranged in $^{n-1}P_{n-1} = (n-1)!$ ways. Now, these two books can be arranged amongst themselves in 2! ways. Hence the total number of permutations in which there two books are placed together is $2!\,(n-1)!$.

The number of permutations of n books without any restriction is $n!$.

Therefore, the number of arrangements in which these two books never occur together

$$= n! - 2!.\,(n-1)! = n.\,(n-1)! - 2.\,(n-1)! = (n-2).\,(n-1)!$$

Example 25. In how many ways can 7 boys and 5 girls be seated in a row so that no two girls may sit together.

Solution. Since there is no restriction on boys, first of all we fix the positions of 7 boys. Their positions are indicated as

$$\times B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times B_6 \times B_7 \times$$

Now 7 boys can be arranged in 7! ways. Now if 5 girls sit at places (including the two ends) indicated by ×, then no two of the 5 girls will sit together. Clearly, 5 girls can be seated in 8 places in $^8P_5$ ways.

Hence the required number of ways of seating 7 boys and 5 girls under the given condition $= {}^8P_5 \times 7!$.

Example 26. In how many ways 4 boys and 4 girls can be seated in a row so that boys and girls are alternate?

Solution.

Case I. When a boy sits at the first place:

Possible arrangements will be of the form

B G B G B G B G

Now there are four places for four boys, therefore four boys can be seated in 4! ways. Again there are four places for four girls, therefore four girls can be seated in 4! ways. Hence the number of ways in this case is $4! \times 4!$

Case II. When a girl sits at the first place:

Possible arrangements will be of the form

G B G B G B G B

Therefore, the number of arrangements in their case is $4! \times 4!$

Hence the required number of ways $= 4! \times 4! + 4! \times 4! = 1152$.

Example 27. Given 10 people $P_1, P_2, P_3, \ldots\ldots, P_{10}$

(*i*) In how many ways can the people be lined up in a row?

(*ii*) How many lineups are there if P_2, P_6, P_9 want to stand together?

(*iii*) How many lineups are there if P_2, P_6, P_9 do not want to stand together?

Solution. (*i*) Without any restriction 10 people can be lined up in a row in 10! ways.

(*ii*) Treating P_2, P_6, P_9 as one, 8 people can be arranged in a row in 8! ways. Again three persons P_2, P_6, P_9 can be arranged among themselves in 3!. So, the number of lined up when particular 3 persons are together is $8! \times 3!$.

(*iii*) The total number of ways in which P_2, P_6, P_9 do not want to stand together = Total number of arrangements without any restriction – Total number of arrangements in which P_2, P_6, P_9 are always together $= 10! - 8! \times 3!$.

4. Formation of numbers with digits

Example 28. How many number of four digits can be formed with the digits 1, 2, 3, 4 and 5, if repetition of digits is not allowed.

Solution. The required number of 4–digit numbers = the number of arrangement of 5 distinct digits taken 4 at a time $= {}^5P_4 = 5 \times 4 \times 3 \times 2 = 120$.

Example 29. How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed.

Solution. Every number lying between 100 and 1000 is a 3–digit number. Therefore, we have to find the number of permutations of 5 digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of 3–digit numbers $= {}^5P_3 = \dfrac{5!}{(5-3)!} = 60$.

Example 30. How many numbers between 400 and 1000 can be formed with the digits 2, 3, 4, 5, 6 and 0, if repetition of digits is not allowed.

Solution. Any number lying between 400 and 1000 must be of 3–digit only. Since the number should be greater than 400, the hundred's place can be filled up by any one of the three digits 4, 5 and 6 in 3 ways. The remaining two places (ten's and unit's) can be filled up by remaining five digits in ${}^5P_2$ ways.

$\therefore$ The required number $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 60$

Example 31. How many four digit numbers are there with distinct digits?

Solution. The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is ${}^{10}P_4$. But these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not 4–digits numbers.

To find the total number of numbers in which 0 is at thousand's place, we fix 0 at thousand's place and arrange remaining 9 digits by taking 3 at a time. The number of such arrangements in ${}^9P_3$. Hence, the total number of 4–digit numbers $= {}^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

5. Word Building

Example 32. Find the number of permutations that can be had from the letters of the DAUGHTER.

(*i*) taking all the letters together.

(*ii*) beginning with D

(*iii*) beginning with D and ending with R

(*iv*) vowels being always together

(*v*) not all vowels together

(*vi*) not even two vowels together

(*vii*) vowels occupying even places.

Solution. The number of letters in the word DAUGHTER is 8 and all are different.

(*i*) Number of words taking all the letters together is equal to the numbers of arrangements of all 8 letters taken all at a time $= 8! = 40320$.

(*ii*) Keeping D fixed in the first place, the remaining seven places can be filled up by the remaining 7 letters in $7! = 5040$ ways.

Hence, the required number of words $= 5040$.

(*iii*) Keeping D fixed in the first place and R in the last place, the remaining 6 places can be filled up by the remaining 6 letters in $6! = 720$ ways.

Hence, the required number of words $= 720$.

(*iv*) Treating the three vowels A, U and E as one, we have $8 - 3 + 1 = 6$ letter to arrange. This can be done in $6! = 720$ ways. Again the three vowels A, U, E can be arranged among themselves in $3! = 6$ ways.

Hence, the required number of words $= 720.6 = 4320$.

(*v*) The required number of words = Total number of all possible words – Number of words with all vowels together.

$= 40320 - 4320 = 36000$.

(*vi*) Since there is no restriction on consonants D, G, H, T, R. We first fix these five consonants. This can be done $5! = 120$ ways.

$$\times C \times C \times C \times C \times C \times$$

The three vowels can be arranged in any of the three of six places marked $\times$, so that no two vowels are together. This can be done in ${}^6P_3 = 6 \times 5 \times 4 = 120$. Hence the required number of permutation $= 120 \times 120 = 14400$.

(*viii*) Out of 8 places, 2nd, 4th, 6th, 8th places are even places and the three vowels A, E, U can be arranged in 4 places in $^4P_3 = 24$ ways.

The four places (1st, 3rd, 5th, 7th) and one remaining even place are to be filled by 5 consonants. This can be done in $5! = 120$ ways.

Hence, the required number of ways $= 24 \times 120 = 2880$.

7.4.3 Permutations of Things not all Different

So far we have considered the situation where all the objects that are to be arranged are different.

Suppose it is required to find the number of ways in which n things may be arranged among themselves, taking all of them at a time, when p of things are alike of one kind, q of them alike of a second kind, r of them of a third kind, and the rest all different.

Let x be the required number of permutations.

Now, replace p alike things by p distinct things which are also different from others. These p different things may be permuted among themselves in $p\,!$ ways without changing the positions of other things.

Thus if p alike things of one kind are replaced by p different things, the number of permutations $= x \times p!$.

Now if in any one of the $x \times p!$ permutations, we replace q alike things of other kind by q other different things, then the number of permutations of these q new things among themselves without disturbing the arrangements of rest will be $q!$.

Thus if both the replacements are done, the number of permutation $= x \times p! \times q!$

Similarly when r alike things are also replaced by r different things, then the number of permutations $= x \times p! \times q! \times r!$.

Now each of these $x \times p! \times q! \times r!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore \quad x \times p! \times q! \times r! = n!$

Hence $x = \dfrac{n\,!}{p\,!q\,!r\,!}$

Formula: The number of permutations that can be formed from a collection of n objects of which n_1 are of one type, n_2 are of a second type, ... n_k are of Kth type with $n_1 + n_2 + \dots + n_k = n$ is

$$\frac{n!}{n_1! \times n_2! \times \dots \times n_k!}$$

Example 33. How many permutations of the letters of the word BANANA?

Solution. There are 6 letters in the word BANANA of which three are alike of one kind (3A's), two are alike of second kind (2N's) and the rest one letter is different.

Hence the required number of permutations $= \dfrac{6\,!}{3\,!2\,!} = 60$

Example 34. Find the number of permutations of the letters of the word CALCULUS.

Solution: The given word has 8 letters, of which C, L and U are repeated twice and 1 each are A and S.

$\therefore$ Required number of permutations $= \dfrac{8!}{2!2!2!1!1!}$

$$= \frac{8!}{8} = 7! = 5040$$

Example 35. How many ways are there to arrange the letters of the word ENGINEERING?

[JNTU(K) May 2018 (Sup.)]

Solution: The given word has 11 letters, of which 3 are E's, 3 are N's, 2 are G's, 2 are I's and 1 is R.

$\therefore$ Required number of ways $= \dfrac{11!}{3!3!2!2!1!}$

$$= \frac{11!}{6\times6\times4} = \frac{11!}{6\times24} = \frac{11!}{6\times4!}$$

$$= \frac{11\times10\times9\times8\times7\times6\times5}{6}$$

$$= 11\times10\times9\times8\times7\times6\times5 = 277200$$

Example 36. Find the number of Permutations of the word MAYAJALAM.

[JNTU (K) Oct. 2019 (Set No. I)]

Solution: The given word has 9 letters, of which 2 are M's, 4 are A's and 1 each of Y, J, L

$\therefore$ Required number of permutations $= \dfrac{9!}{2!.4!.1!.1!.1!} = \dfrac{9!}{2\times4!}$

$$= \frac{9\times8\times7\times6\times5}{2} = 9\times8\times7\times3\times5$$

$$= 7560$$

Example 37. Obtain the number of permutations of all the letters of the words:
(*i*) COMMITTEE (*ii*) ACCOUNTANT. *[JNTU(A) (R15) Aug. 2021]*

Solution: (*i*) The given word has 9 letters, of which 2 are M, T, E and 1 each are C, O, I

$\therefore$ Required no. of permutations $= \dfrac{9!}{2!2!2!1!1!1!} = \dfrac{9!}{8} = 9\times7\times6\times5! = 63\times6\times120 = 45,360$

(*ii*) This is left as an exercise to the reader.

Example 38. How many arrangements can made out of the letters of the word 'Mathematics'

[JNTU(A) Nov. 2018]

Solution: The given word has 11 letters, of which there as 2M' s, 2A' s, 2T' s, and 1 each of H, E, I, C and S.

$\therefore$ The required no. of permutations $= \dfrac{11!}{2!2!2!1!1!1!1!1!}$

$$= \frac{11!}{8} = 11\times10\times9\times7\times6\times5! = 4,989,600$$

Example 39. Find the number of possible ways in which the letters of the word COTTON can be arranged so that the two Ts don't come together.

Solution. There are two T's, two O's and the rest two letters are different in the word COTTON. Hence the number of arrangements of the letters without any restriction

$$= \frac{6!}{2!2!} = 180$$

Considering the two T's as one letter, the number of letters to be arranged is 5.

So, the number of arrangements in which both T come together $= \frac{5!}{2!} = 60$

Hence, the number of ways in which two T's do not come together $= 180 - 60 = 120$.

Example 40. A coin is tossed 6 times. In how many ways can we obtain 4 heads and 2 tails?

Solution. Whether we toss a coin 6 times or toss 6 coins at a time, the number of arrangements will be the same.

$\therefore$ The number of arrangements of 4 heads and 2 tails out of 6 is $\frac{6!}{4!2!} = 15$.

Example 41. There are 3 copies each of 4 different books. Find the number of ways of arranging them on a shelf.

Solution. Total number of books is $3 \times 4 = 12$

Each of the 4 different titles has 3 copies each.

$\therefore$ The required number of ways of arranging them on a shelf is

$$\frac{12!}{3!3!3!3!} = \frac{12!}{(3!)^4} = 369600.$$

7.4.4 Permutations with repetitions allowed

The number of permutations of n different things taken r at a time, when things may be repeated any number of times is n^r.

Suppose r places are to be filled with n things. The first place can be filled up in n ways and when it has been filled up in any one of these ways, the second place can also be filled up in n ways for the thing occupying the first place may occupy the second place also. Thus the first two places can be filled in $n \times n = n^2$ ways. When the two places are filled in any of the n^2 ways, the third place can also be filled up in n ways, and so the three places together can be filled up in $n^2 \times n = n^3$ ways.

Proceeding in this way, we conclude that the r places can be filled in $n \times n \times n \times \ldots\ldots r$ times *i.e.* n^r ways.

Example 42. How many numbers of 3 digits can be formed with the digits 2, 4, 6 and 8 when a digit may be repeated any number of times?

Solution. Since repetition is allowed, each of the 3 places in a 3–digit numbers can be filled up in 4 ways by the given 4 digits.

hundreds place	*tens place*	*units place*
4 ways	4 ways	4 ways

$\therefore$ The required number of 3–digit numbers $= 4^3 = 64$.

Example 43. In how many ways 3 prizes can be distributed among 4 boys when a boy can get any number of prizes.

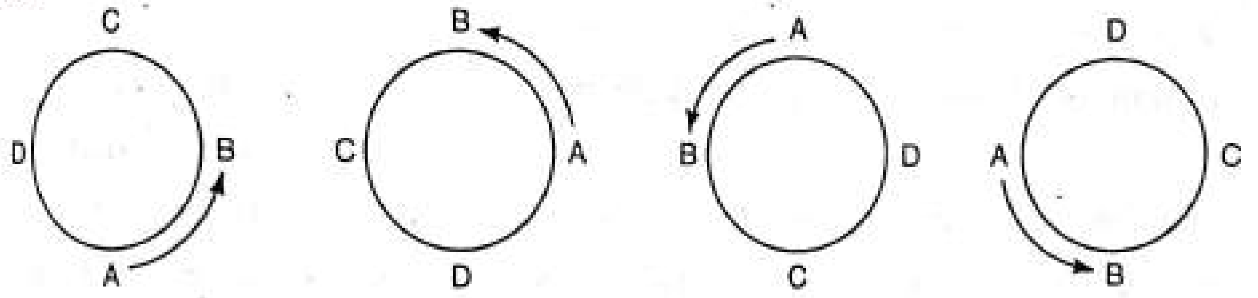
Solution. Each of the three prizes can be given in 4 ways since a boy can receive any number of prizes.

$\therefore$ The number of ways in which all the prizes can be given $= 4 \times 4 \times 4 = 4^3 = 64$.

7.4.5 Circular Permutation

The permutation that has been described above are more properly called linear permutations as the objects are being arranged in a line (or row). If we arrange them in a circle, then the number of permutations change. A circular permutation is an arrangements of objects around a circle.

Consider four persons A, B, C and D who are to be arranged around a circle. The following arrangements are one and the same as relative position of none of the four positions A, B, C, D is changed.



(i) (ii) (iii) (iv)

But in case of linear arrangements the four arrangements are different.

A B C D   D A B C   C D A B   B C D A

Thus the circular permutations are different only when the relative order of the objects is changed otherwise they are the same.

As the number of circular permutations depend on the relative positions of objects, the number of permutation of n objects in a circle can be found by keeping one element fixed and arranging remaining $(n-1)$ elements which can be done in $(n-1)!$ ways.

$$\boxed{\text{Thus number of circular arrangements of } n \text{ different things} = (n-1)!}$$

Example 44. In how many ways 5 boys and 4 girls can be seated at a round table if

(i) there is no restriction

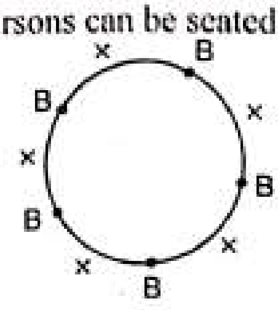
(ii) all the four girls sit together

(iii) all the four girls do not sit together

(iv) no two girls sit together.

Solution: (i) Total number of persons $= 5 + 4 = 9$. These persons can be seated at the round table in $(9-1)! = 8!$ ways.

(ii) Taking 4 girls as one person, we have $5 + 1 = 6$ persons. These 6 persons can be seated at a round table in $(6-1)! = 5!$ ways. But 4 girls can be arranged among themselves in 4! ways. So the required number of ways $= 5! \times 4!$.



(iii) The number of arrangements when all the 4 girls do not sit together = total number of arrangements without restriction − number of arrangements when all the four girls sit together $= 8! - 5! \times 4!$.

(*iv*) Since there is no restriction on boys, first of all we arrange 5 boys to seat at round ta and this can be done $(5-1)! = 4!$ ways. Now there are 5 places for 4 girls as indicated by Therefore, 4 girls can be seated in ${}^5P_4 = 5!$ ways.

So, the required number of ways $= 4! \times 5!$

Example 45. In how many ways can a party of 4 boys and 4 girls be seated at a circular ta so that no 2 boys are adjacent.

Solution: 4 girls can be seated at a circular table in 3! ways. When they have been seat there remain 4 places for the boys each between two girls. Therefore, the boys can sit in ways. Thus, there are $3! \times 4!$ *i.e.* 144 ways of seating in the party.

Example 46. How many ways are there to seat 10 boys and 10 girls around a circular ta if boys and girls seat alternatively? [*JNTU(K) Oct. 2019 (Set No.*

Solution: 10 girls can be seated at a circular table in $(10-1)! = 9!$ ways. After the girls ha been seated, there remain 10 places for boys each between two girls. Thus, the boys can sit 10! Ways. Hence, the required number $= 9! \times 10!$.

Example 47. In how many ways can 5 children arrange themselves in a ring.

[*JNTU(A) (R19)* March 202

Solution: The number of ways of arranging 5 children in a ring $= (5-1)! = 4! = 24$

Example 48. 15 males and 10 females are seated in a round table meeting. How many wa they can be seated if all the females are to be seated together? [*JNTU(A) (R19)* March 2021

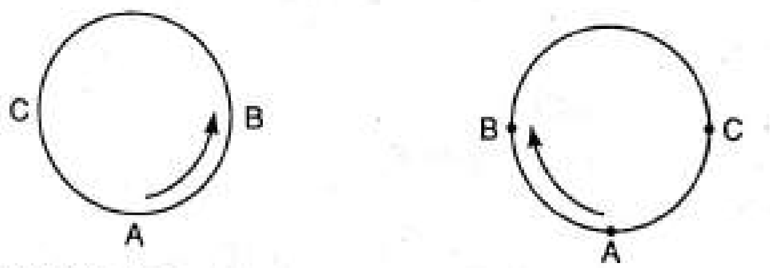
Solution: Let us first seat the 15 males by treating 10 females as one unit. Then 15 + 1 16 Persons can be seated at a round table in $(16-1)! = 15!$ ways. But the 10 females can arranged themselves in 10! Ways.

Hence, the required number of ways $= 15! \times 10!$

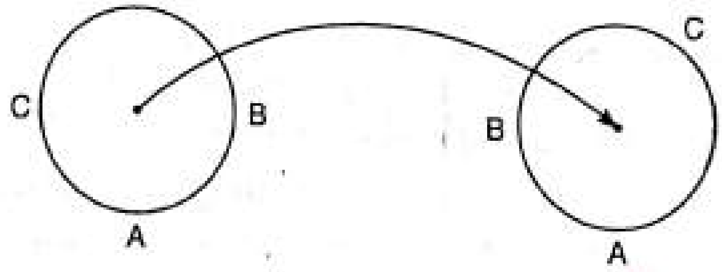
7.4.6 Clockwise and Counter-clockwise Permutations

There are two types of circular permutations

(*i*) Those permutations in which clockwise and anti-clockwise arrangements a distinguishable. Thus while seating 3 persons A, B, C around a table, the following permutatio are different.



(*ii*) When clockwise and anti-clockwise permutations are not different.



When clockwise and anti-clockwise arrangements are not different *e.g.*, arrangement of beads in a necklace, arrangements of flowers in a garland etc, number of circular arrangements of n different things is $\frac{1}{2}(n-1)!$.

Example 49. In how many ways 8 different beads can be arranged to form a necklace?

Solution. Eight different beads can be arranged in circular form in $(8-1)! = 7!$ ways. Since there is no distinction between the clockwise and anti-clockwise arrangements, the required number of arrangements $= \frac{1}{2}(7!)$.

Example 50. In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements?

Solution. Seven persons can sit at a round table in $(7-1)! = 6!$ ways. But, in clockwise and anti-clockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= \frac{1}{2}(6!) = 360$.

7.5 COMBINATIONS

The different groups or selections that can be made out of a given set of things by taking some or all of them at a time irrespective of the order are called their combinations.

The number of combinations of n different things taken $r(\leq n)$ at a time is denoted by $C(n, r)$ or ${}^nC_r$ (or) $\binom{n}{r}$. For example,

The selection of two letters from three letters a, b, c are ab bc ca and thus, the number of combinations of 3 letters taken 2 at a time is $C(3, 2) = 3$

The number of combinations of 4 letters a, b, c, d taken two at a time is $C(4, 2) = 6$ and they are ab, ac, ad, bc, bd, cd.

7.5.1 Differences Between a Permutation and Combination

1. In a combination only selection is made where in a permutation not only a selection is made but also an arrangement in a definite order is considered.

2. In a combination, the ordering of the selected objects is immaterial. In a permutation, this ordering is essential. For example, a, b and b, a are same in combination but they are different in permutation.

Generally we use the word arrangements for permutations and the word selections for combinations.

7.5.2 The value of $C(n, r)$

To find the number of combination of n distinct objects taken r $(\leq n)$ of a time.

Let x be the required number of combinations *i.e.*, $x = C(n, r)$. Now each combination contains r different things which can be arranged among themselves in $r!$ ways. Hence x such combinations give rise to $x \times r!$ arrangements giving the number of permutations of n different things taken r at a time *i.e.*, $P(n, r)$

$\therefore \qquad x \times r! = P(n, r) = \dfrac{n!}{(n-r)!}$

$\therefore \qquad x = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$

or $\qquad C(n, r) = \dfrac{n!}{r!(n-r)!}$ for $0 \le r \le n$

Cor 1. $C(n, n) = C(n, 0) = 1$

Putting $r = n$ in (1) $C(n, n) = \dfrac{n!}{n!(n-n)!} = \dfrac{1}{0!} = 1$

Putting $r = 0$ in (1) $C(n, 0) = \dfrac{n!}{0!(n-0)!} = \dfrac{n!}{0!\,n!} = 1$

Note: $(r!)\,{}^nC_r = {}^nP_r$

7.5.3 Properties of ${}^nC_r$ or $C(n, r)$

Property 1. ${}^nC_r = {}^nC_{n-r}\ (0 \le r \le n)$.

We have ${}^nC_{n-r} = \dfrac{n!}{(n-r)!(n-n+r)!} = \dfrac{n!}{(n-r)!\,r!} = {}^nC_r$

Property 2. ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\ (0 \le r \le n)$.

We have

$$\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{[n-(r-1)]!\,(r-1)!} \\
&= n!\left[\frac{1}{(n-r)!\,r!} + \frac{1}{(n-r+1)!\,(r-1)!}\right] \\
&= n!\left[\frac{1}{(n-r)!\,r(r-1)!} + \frac{1}{(n-r+1)\,(n-r)!\,(r-1)!}\right] \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right] \\
&= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n+1}{r\,(n-r+1)}\right] \\
&= \frac{(n+1)\,n!}{r\,(r-1)!\,(n-r+1)\,(n-r)!} \\
&= \frac{(n+1)!}{r!\,(n-r+1)!} \\
&= {}^{n+1}C_r.
\end{aligned}$$

$\therefore \qquad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

Property 3. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$.

We have ${}^nC_x = {}^nC_y$

$\Rightarrow {}^nC_x = {}^nC_{n-y}\ [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow x = y$ or $x = n - y$

$\Rightarrow x = y$ or $x + y = n$.

Relation between ^nC_r and $^{n+1}C_r$.

$$\frac{^{n+1}C_r}{^nC_r} = \frac{(n+1)!/r!(n-r+1)!}{n!/r!(n-r)!} = \frac{(n+1)!}{n!} \times \frac{(n-r)}{(n-r+1)!} = \frac{n+1}{n-r+1}$$

Example 51. Evaluate the following

(i) $^6C_3$ (ii) $^{10}C_8$

Solution. (i) $^6C_3 = \frac{6!}{3!(6-3)!}\left[\because {^nC_r} = \frac{n!}{r!(n-r)!}\right]$

$$= \frac{6 \times 5 \times 4 \times 3!}{3!(6-3)!} = \frac{120}{3!} = 20$$

(ii) $^{10}C_8 = {^{10}C_2}$ $[\because {^nC_r} = {^nC_{n-r}}]$

$$= \frac{10!}{(10-2)!2!} = \frac{10 \times 9 \times 8!}{8! \times 2!} = 45.$$

Example 52. Compute 8_{P_5} and 6_{C_3}. **[JNTU(K) May 2019 (Sup.)]**

Solution: $8_{P_5} = \frac{8!}{(8-5)!}\left[\because {^nP_r} = \frac{n!}{(n-r)!}\right] = \frac{8!}{3!} = 8 \times 7 \times 6 \times 5 \times 4 = 6720$

$$6_{C_3} = \frac{6!}{3!(6-3)!}\left[\because {^nC_r} = \frac{n!}{r!(n-r)!}\right] = \frac{6!}{3!3!} = \frac{6 \times 5 \times 4}{6} = 20$$

Example 53. *If* $^{18}C_r = {^{18}C_{r+2}}$, *find the value of* $^rC_5$.

Solution. As $r \neq r + 2$ so $r + (r+2) = 18 \Rightarrow r = 8$

$$\therefore {^rC_5} = {^8C_5} = {^8C_3} = \frac{8 \times 7 \times 6}{3!} = 56.$$

Example 54. *If* $^nC_x = 56$ *and* $^nP_x = 336$, *find n and x*

Solution. Here $^nP_x = 336$ *i.e.,* $\frac{n!}{(n-x)!} = 336$

and $^nC_x = 56$ *i.e.,* $\frac{n!}{x!(n-x)!} = 56$

$$\Rightarrow \frac{n!}{x!(n-x)!} \times \frac{(n-x)!}{n!} = \frac{56}{336}$$

$$\Rightarrow \frac{1}{x!} = \frac{1}{6} = \frac{1}{3!}$$

$\therefore x = 3.$

Again $^nP_3 = 336$ *i.e.,* $\frac{n!}{(n-3)!} = 336$

or $\frac{n(n-1)(n-2)(n-3)!}{(n-3)!} = 336$

or $n(n-1)(n-2) = 336 = 8 \times 7 \times 6$

or $\quad n(n-1)(n-2) = 8 \times (8-1) \times (8-2)$

$\therefore \quad n = 8$

Example 55. If ${}^{1000}C_{98} = {}^{999}C_{97} + {}^{x}C_{901}$; find x.

Solution. Here ${}^{1000}C_{98} = {}^{999}C_{97} + {}^{x}C_{901}$

or $\quad {}^{1000}C_{902} = {}^{999}C_{902} + {}^{x}C_{901}$ $(\because {}^{n}C_{r} = {}^{n}C_{n-r})$

or $\quad {}^{999+1}C_{902} = {}^{999}C_{902} + {}^{x}C_{902-1}$

As $\quad {}^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1}$

we get $x = 999$.

Example 56. In how many ways can 4 questions be selected from 7 questions?

Solution. The required number of ways $= {}^{7}C_{4} = {}^{7}C_{3} = \dfrac{7 \times 6 \times 5}{3.2.1} = 35.$

Example 57. In how many ways can you select at least one king, if you choose five cards from a Deck of 52 cards? *[JNTU (K) May 2018 (Sup.), (A)(R19) March 2021]*

Solution: There are 4 kings in a deck of 52 cards.

The number of ways of choosing 1 king $= 4_{C_1} \times 48_{C_4}$

The number of ways of choosing 2 kings $= 4_{C_2} \times 48_{C_3}$

The number of ways of choosing 3 kings $= 4_{C_3} \times 48_{C_2}$

The number of ways of choosing 4 kings $= 4_{C_4} \times 48_{C_1}$

$\therefore$ The required number $= 4_{C_1} \times 48_{C_4} + 4_{C_2} \times 48_{C_3} + 4_{C_3} \times 48_{C_2} + 4_{C_4} \times 48_{C_1}$

$= 886,656$

Example 58. If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?

Solution. When two persons shake hands, it is counted as one handshake, not two. Therefore, the total number of handshakes is the same as number of ways of selecting 2 persons from among 12 persons. This can be done in

$${}^{12}C_2 = \frac{12 \times 11}{2 \times 1} = 66 \text{ ways.}$$

Example 59. There are 35 students and 4 teachers. In how many ways every student shake hand with other students and all the teachers. *[JNTU(K) May 2019 (Sup), (R19) March 2021]*

Solution: When two persons shake hands, it is counted as one handshake. Therefore, the total number of handshakes is the same as number of ways of selecting 2 students from among 35 students. This can be done in $35_{C_2} = \dfrac{35!}{2!33!} = \dfrac{35 \times 34 \times 33!}{2 \times 33!} = 35 \times 17 = 595$

The number of ways of every student shakes hand with 4 teachers $= 35 \times 4 = 140$

Hence, required number $= 595 + 140 = 735$

Example 60. A group of 8 scientists is composed of 5 - psychologists and 3 - sociologists. In how Many ways can a committee of 5 be formed that has 3 - psychologists and 2 - sociologists?

[JNTU (A) (R19) Aug. 2021 (Sup)]

Solution: The desired number $= {}^5C_3 \times {}^3C_2 = \dfrac{5!}{3!2!} \times \dfrac{3!}{2!1!}$

$$= \frac{5 \times 4}{2} \times 3 = 10 \times 3 = 30$$

Example 61. In how many ways can a committee of 5 teachers and 4 students be chosen from 9 teachers and 15 students?

Solution. The four students can be chosen out of 15 students in ${}^{15}C_4$ and the nine teachers can be selected in ${}^9C_5$ ways. Now each way of selecting students can be associated with each of selecting teachers.

So, the required number ways of selecting committee $= {}^{15}C_4 \times {}^9C_5$.

Example 62. Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can this be done so as to include (i) exactly one woman (ii) at least one woman

Solution. (*i*) We have to select one woman out of 2 and 2 men out of 5.

The number of ways of selecting 1 woman $= {}^2C_1 = 2$

The number of ways of selecting 2 men $= {}^5C_2 = \dfrac{5.4}{2} = 10$

$\therefore$ The committee can be formed in $2 \times 10 = 20$ ways.

(*ii*) The committee can be formed in the following ways

(*a*) 2 men and 1 woman.

(*b*) 1 man and 2 women.

For (*a*) 2 men out of 5 men and 1 woman out of 2 women can be selected in ${}^5C_2 \times {}^5C_1$ ways.

For (*b*) 1 man out of 5 men and 2 women out of 2 women can be selected in ${}^5C_1 \times {}^2C_2$ ways.

$\therefore$ Total number of ways of forming the committee $= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25$.

Example 63. In how many ways can a student choose a course of 5 subjects if 9 subjects are available and 2 subjects are compulsory for all the students.

Solution. Since two subjects are compulsory, the students are required to choose 3 out of 7 subjects and this can be done in ${}^7C_3 = \dfrac{7!}{3!4!} = 35$ ways.

Example 64. The question paper of mathematics contains ten questions divided into two groups of five questions each. In how many ways can an examinee answer six questions taking at least two questions from each group.

Solution. The examinee can answer questions from two groups in the following ways.

(*i*) 2 from first group and 4 from the second group.

(*ii*) 3 from first group and 3 from the second group.

(*iii*) 4 from first group and 2 from the second group.

For (*i*), the number of ways of selecting the questions $= {}^5C_2 \times {}^5C_4 = 10 \times 5 = 50$

For (*ii*), the number of ways of selecting the questions $= {}^5C_3 \times {}^5C_3 = 10 \times 10 = 100$

For (*iii*), the number of ways of selecting the questions $= {}^5C_4 \times {}^5C_2 = 5 \times 10 = 50$

Hence the examinee can answer the questions in either of the three ways.

Therefore, the required number of ways $= 50 + 100 + 50 = 200$.

Example 65. A women has 20 close relatives and she wishes to invite 7 of them to dinner. In how many ways she can invite them in the following situations:

(*i*) There is no restriction on the choice.

(*ii*) Two particular persons will not attend separately

(*iii*) Two particular persons will not attend together. **[JNTU(K) *Oct. 2017*(Set No. 4)]**

Solution: (*i*) Since there is no restriction on the choice of invites, required number of ways of inviting 7 out of 20

$$= {}^{20}C_7 = \frac{20!}{7!13!} = \frac{20\times19\times18\times17\times16\times15\times14}{7\times6\times5\times2\times1}$$

$$= \frac{390,700,800}{420} = 930\ 240 \text{ ways}$$

(*ii*) Two particular persons will not attend separately means that they should both be invited or not invited.

(*a*) Suppose both of them are invited. Then we have to select 5 invites from remaining 18 relatives. This can be done in ${}^{18}C_5 = \frac{18!}{5!13!} = \frac{18\times17\times16\times15\times14}{120} = \frac{1028160}{120} = 8568$ ways.

(*b*) Suppose both of them are not invited. Then, seven invitees are to be selected from 18 relatives. This can be done in ${}^{18}C_7 = \frac{18!}{7!11!} = \frac{18\times17\times16\times15\times14\times13\times12}{5040} = 31,824$ ways.

Hence, total number of ways in which the invitees can be selected $= 8568 + 31824 = 40392$

(*iii*) Suppose two particular persons, say A and B will not attend together. So, only one of them can be invited or none of them can be invited.

(*a*) Suppose A is invited. Therefore, the required number $= {}^{18}C_5 = \frac{18!}{5!13!} = 8568$

(*b*) Suppose B is invited. Then, the required number $= {}^{18}C_5 = 8568$

(*c*) If both A and B are not invited. Then, the number of ways of choosing the invitees

$$= {}^{18}C_7 = \frac{18!}{7!11!} = 31824$$

Hence, total number of ways in which the invitees can be selected $= 8568 + 8568 + 31824$
$= 48960$

Example 66. There are 50 students in each of the senior or junior classes. Each class has 25 male and 25 female students. In how many ways can an eight student committee be formed so that there are four females and three junior in the committee?

Solution. A committee of 8 students can be formed in the following ways.

	Senior (50)		*Junior (50)*	
	Male (25)	*Female (25)*	*Male (25)*	*Female (25)*
(*i*)	1	4	3	–
(*ii*)	2	3	2	1
(*iii*)	3	2	1	2
(*iv*)	4	1	–	3

For (i), the number of ways of selecting 8 students = ${}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3$.

For (ii), the number of ways of selecting 8 students = ${}^{25}C_2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1$.

For (iii), the number of ways of selecting 8 students = ${}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2$.

For (iv), the eight students can be selected in ${}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3$.

Since the committee of 8 students is formed in each case, the total number of ways of forming the committee

$$= {}^{25}C_1 \times {}^{25}C_4 \times {}^{25}C_3 + {}^{25}C_2 \times {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 + {}^{25}C_3 \times {}^{25}C_2 \times {}^{25}C_1 \times {}^{25}C_2 + {}^{25}C_4 \times {}^{25}C_1 \times {}^{25}C_3.$$

Example 67. Find the number of diagonals that can be drawn by joining the angular points of hexagon.

Solution. A hexagon has six angular points and six sides. The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points $= {}^6C_2 = \dfrac{6 \times 5}{2} = 15$

But the number of sides is 6, hence the number of diagonals = 15 – 6 = 9.

Example 68. Prove the identity $C(n+1, r) = C(n, r-1) + C(n, r)$

[JNTU(K) Oct.2017 (Set No. 3)]

Solution: R.H.S $= C(n, r-1) + C(n, r)$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \left[\because C(n,r) = \frac{n!}{r!(n-r)!}\right]$$

$$= \frac{n!}{(r-1)!(n-r+1)(n-r)!} + \frac{n!}{r(r-1)!(n-r)!} [\because n! = n(n-1)!]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!} \cdot \frac{n+1}{r(n-r+1)}$$

$$= \frac{(n+1)n!}{r(r-1)!(n-r+1)(n-r)!} + \frac{(n+1)!}{r!(n-r+1)!} [\because n! = n(n-1)!]$$

$$= C(n+1, r) = \text{L.H.S}$$

Example 69. A certain question paper contains two parts A and B each containing 4 questions. How many different ways a student can answer 5 questions by selecting at least 2 questions from each part? *[JNTU(A)(R15) March 2021]*

Solution: The possibilities for a student can select 5 questions are

(i) 3 questions from Part A and 2 questions from Part B. This can be done in $C(4, 3) \times C(4, 2)$

$$= \frac{4!}{3!\,1!} \times \frac{4!}{2!\,2!} = \frac{24}{6} \times \frac{24}{4} = 4 \times 6 = 24 \text{ ways}$$

(ii) 2 questions from Part A and 3 questions from Part B. This can be done in $C(4, 2) \times C(4, 3)$ = 24 ways

∴ The total number of ways a student can answer his 5 questions = 24 + 24 = 48.

Example 7. A committee of eight is to be formed from 16 men and 10 women. In how many ways can the committee be formed, if

(*i*) There are no restrictions?

(*ii*) There must be 4 men and 4 women

(*iii*) An even number of women

(*iv*) At least 6 men

[JNTU (A) Nov 20..]

Solution:

(*i*) Since there are no restrictions, therefore the required number $= C(26, 8) = \frac{26!}{8!18!}$

(*ii*) 4 men and 4 women $= C(16, 4). C(10, 4)$

$$= \frac{16!}{4!12!} \times \frac{10!}{4!6!} = \frac{16\times15\times14\times13}{24} \times \frac{10\times9\times8\times7}{24}$$

$$= 1820 \times 210 = 382{,}200$$

(*iii*) (*a*) 6 men and 2 women $= C(16, 6). C(10, 2)$

(*b*) 4 men and 4 women $= C(16, 4). C(10, 4)$

(*c*) 2 men and 6 women $= C(16, 2). C(10, 6)$

(*d*) no men and 8 women $= C(16, 0). C(10, 8) = C(10, 8)$

$\therefore$ The total number of possible ways $= C(16, 6). C(10, 2) + C(16, 4). C(10, 4) + C(16, 2). C(10, 4) + C(10, 8)$

(*iv*) (*a*) 6 men and 2 women $= C(16, 6). C(10, 2)$

(*b*) 7 men and 1 women $= C(16, 7). C(10, 1)$

(*c*) 8 men and 0 women $= C(16, 8). C(10, 0) = C(16, 8)$

$\therefore$ The total number of possible ways $= C(16, 6). C(10, 2) + C(16, 7). C(10, 1) + C(16, 8)$

7.5.4 Combinations Taken Any Number at a Time

Total number of combinations of n different things taken any number at a time is $2^n - 1$. Each thing may be disposed of in two ways. It may be either accepted or rejected. Corresponding to each of the two ways of disposing the first, the second can be disposed of in the similar two ways. So, the total number of ways of disposing of all the things $= 2 \times 2 \times$ ......n times $= 2^n$. But in this 2^n cases, there is one case in which all the things are rejected. Hence the total number of ways in which one or more things are taken $= 2^n - 1$.

Example 71. You have 4 friends; in how many ways can you invite one or more them for dinner?

Solution. You may invite 1, 2, 3 or 4 of your friends to dinner. Hence the required number of ways $= 2^4 - 1 = 63$.

Example 72. There are 5 questions in a question paper. In how many was can a boy solve one or more questions.

Solution. The boy may either solve a questions or leave it. Thus the boy disposes of each question in two ways. Thus the number of ways of disposing of all the questions $= 2^5$. But this includes the case in which he has left all the questions unsolved. Hence the required number of ways of solving the paper $= 2^5 - 1 = 31$.

7.5.5 Combinations of Things Not all Different

By taking some or all of $(p + q + r + ...)$ things, where p are alike of one kind, q are alike of a second kind, r alike of third kind and so on, the total number of possible selection is

$$[\{(p + 1)(q + 1)(r + 1) ..... \} - 1].$$

Out of p things we may take 0, 1, 2, 3, ... or p. Hence they can be disposed of in $(p + 1)$ ways. Similarly, q alike things may be disposed of in $(q + 1)$ ways; r alike things in $(r + 1)$ ways and so on. Since each way of disposing p like things can be associated with each way of disposing any other thing, the total number of ways in which all the things can be disposed of is $(p + 1)(q + 1)(r + 1)$ .... But this includes the case in which all the things are left out. Rejecting this, the required number of combinations is $[\{(p + 1)(q + 1)(r + 1) ......\} - 1]$.

Example 73. In how many ways can a selection be made out of 3 mangoes, 5 oranges and 2 apples?

Solution. Hence $p = 3, q = 5$ and $r = 2$.

So, the required number of combinations $= (3 + 1)(5 + 1)(2 + 1) - 1$

$$= 4.6.3 - 1 = 72 - 1 = 71.$$

7.5.6 Division Into Groups

To find the number of ways in which $(p + q)$ things can be divided into two groups containing p and q things respectively.

We can select p things from $(p + q)$ things in ${}^{p+q}C_p$ ways, and each time p things are taken, a group of q things are left aside from which q things can be selected in ${}^{q}C_q$, *i.e.*, 1 way. Again if q things are selected first from $p + q$ things in ${}^{p+q}C_q$ ways, everytime q things are selected, there remain p things which all can be selected in ${}^{p}C_p$, *i.e.*, 1 way.

As ${}^{p+q}C_p = {}^{p+q}C_{(p+q)-p} = {}^{p+q}C_q = \dfrac{(p+q)!}{p!q!}$, the required number of ways $= \dfrac{(p+q)!}{p!q!}$

Note. If $p = q$, the groups are equal and here it is possible to interchange the two groups without obtaining a new distribution. Hence the number of different ways of division into groups

$$= \frac{(2p)!}{2p!p!} = \frac{(2p)!}{2!(p!)^2}.$$

If the two groups are different, then the number of different ways of division into groups

$$= \frac{(2p)!}{(p!)^2}.$$

Example 74. In how many ways 14 oranges can be divided equally into 2 groups?

Solution. The required number of ways $= \dfrac{14!}{2!7!7!}$.

Example 75. In how many ways 14 oranges can be divided equally among 2 children?

Solution. The number ways of different division $= \dfrac{14!}{(7!)^2}$ for, in this case a group of 7 oranges alloted for the first boy, if given to the second boy, the two groups are different.

7.5.7 Generalization

The number of ways in which $p+q+r$ things can be divided into three groups containing p, q and r things respectively

$$= {}^{p+q+r}C_p \times {}^{q+r}C_q \times {}^{r}C_r = \frac{(p+q+r)!}{p!(q+r)!} \times \frac{(q+r)!}{q!r!} \times 1 = \frac{(p+q+r)!}{p!q!r!}$$

Similarly the result can be extended to the case of dividing a given number of things into more than three groups.

Note 1. The number of ways in which $3p$ things can be divided equally into three distinct groups is $\frac{(3p)!}{(p!)^3}$ $(q = p, r = p)$

Note 2. The number of ways in which $3p$ things can be divided into three **identical** groups is $\frac{(3p)!}{3!(p!)^3}$.

Example 76. In how many ways can 14 men be partitioned into 6 teams where the first team has 3 members, the 2nd team has 2 members, the 3rd team has 3 members, 7th, 4th, 5th, 6th teams each has 2 members?

Solution. The required number is $\frac{14!}{3!2!3!(2!)^4}$

Note 1. The total number of ways to divide n identical things among r persons $= {}^{n+r-1}C_{r-1}$

Note 2. The total number of ways to divide n identical things among r persons so that each gets at least one $= {}^{n-1}C_{r-1}$

Example 77. How many ways can we distribute 14 indistinguishable balls in 4 numbered boxes so that each box is non-empty. *[JNTU(A) (R19) Aug. 2021]*

Solution: The desired number $= {}^{n-1}C_{r-1} = {}^{13}C_3$

$$= \frac{13!}{3!10!} = \frac{13 \times 12 \times 11}{6} = 286$$

7.5.8 Combinations With Repetitions

We first consider the following example:

Example 78. Consider the set {a b c d}. In how many ways can we select two of these letters when repetition is allowed.

Solution. If order matters and repetition is allowed, there are $2^4 = 16$ possible selections and they are

aa	*ba*	*ca*	*da*
ab	*bb*	*cb*	*db*
ac	*bc*	*cc*	*dc*
ad	*bd*	*cd*	*dd*

If order does not mater but repetitions are allowed, there are 10 possibilities and these possibilities are *aa* *bb* *cc* *dd*

ab *bc* *cd* *ad* *ac* *bd*

Theorem 6.5. The number of unordered choices of r from n, with repetitions allowed is

$$\binom{n+r-1}{r} = C(n+r-1, r)$$

Proof : Any choice will consist of x_1 choices of the first object, x_2 choices of the second object, and so on, subject to the condition $x_1 + ... + x_n = r$. So the required nimber is just the number of solutions of the equation $x_1 + ... + x_n = r$ in non-negative integers x_i.

Now we can represent a solution $x_1, ..., x_n$ by a binary sequence:

x_1 0s, 1, x_2 0s, 1, x_3 0s, 1, ..., 1, x_n 0s.

Think of the 1 s as indicating a move from one object to the next. For example, the solution $x_1 = 2, x_2 = 0, x_3 = 2, x_4 = 1$ of $x_1 + x_2 + x_3 + x_4 = 5$ corresponds to the binary sequence 00110010. Two consecutive 1's mean that there are no objects in that place. Corresponding to $x_1 + x_2 + ... + x_n = r$, there will be $n - 1$, 1s and r 0s, and so each sequence will be of length $n + r - 1$, containing exactly r 0s. Conversely, any such sequence corresponds to a non-negative integer solution of $x_1 + x_2 + ... + x_n = r$. Now the r 0s can be in any of the $n + r - 1$ places, so the number of such sequences, and hence the number of unordered choices, is $C(n + r - 1, r)$, the number of ways of choosing r places out of $n + r - 1$.

This proof also establishes the following result.

Theorem 6.6. The number of solutions of $x_1 + x_2 + ... + x_n = r$ in non-negative integers x_i is $C(n + r - 1, r)$.

The equivalent formulation of the above theorem makes it easier to conceptualize the problem.

The number of non-negative integral solutions of $x_1 + x_2 + ... + x_n = r$ is equal to the number of ways of placing r identical (indistinguishable) balls in n numbered boxes.

At this point, we recognize the equivalence of the followings:

The number of r-combinations of n distinct objects with unlimited repetitions

= the number of non-negative integral solutions (*i.e.* $x_1 \geq 0, x_2 \geq 0, ..., x_n \geq 0$) of

$$x_1 + x_2 + ... + x_n = r$$

= the number of ways of distributing r similar balls into n numbered boxes

= the number of binary numbers with $n - 1$ one's and r zeros

$= C(n - 1 + r, r) = C(n + r - 1, r) = C(n - 1 + r, n - 1)$

$= \frac{(n+r-1)!}{[r!(n-1)!]}$.

Note: (*i*) In terms of distributions, this is valid only when the r objects being distributed are identical and the n boxes are distinct.

(*ii*) If we want to ensure that every box gets at least one ball, we must give a ball to each box, then distribute the remaining $r - n$ balls in n boxes which can be done in $C(n - 1 + r - n, n - 1) = C(r - 1, n - 1)$ ways.

(*iii*) The number of positive integral solutions of $x_1 + x_2 + ... + x_n = r$ where $x_1 \geq 1, x_2 \geq 1, ..., x_n \geq 1$ is the same as the number of ways to distribute n identical things among r persons each getting at least $1 = C(r - 1, n - 1)$

(iv) The number of non-negative integral solution of $x_1 + x_2 + ... + x_n = r$ where $x_1 \geq r_1, x_2 \geq r_2, ..., x_n \geq r_n$ is

$= C(n - 1 + r - r_1 - r_2 ... r_n, r - r_1 - r_2, ... r_n)$

$= C(n - 1 + r - r_1 - r_2, ... r_n, n - 1)$

(v) The number of ways of distributing n chaklets to r children so that each child get at least one is $C(n - 1, r - 1)$

(vi) The number of non – negative integer solutions of $x_1 + x_2 + ... + x_r = n$ $(x_i \geq 0)$ is $C(n - 1, n - r)$.

(vii) $C(n + r - 1, r)$ represents the number of ways in which r identical objects can be distributed among n distinct containers

(viii) The number of non–negative integer solutions of the equation

$x_1 + x_2 + ... + x_n = r$ is $C(n + r - 1, r)$

Example 79. In how many different ways can 20 identical apples be distributed among 4 persons?

Solution. The required number $= {}^{(n+r-1)}C_r$

$= {}^{20+4-1}C_{4-1}$ $(\because r = 20, n = 4)$

$= {}^{23}C_3 = \dfrac{23 \times 22 \times 21}{3!} = 1771$

Example 80. Find the number of ways to distribute 20 identical balls in 4 different boxes so that no box remains empty.

Solution. The required number $= {}^{(r-1)}C_{n-1}$

$= {}^{20-1}C_{4-1}$ $(\because r = 20, n = 4)$

$= {}^{19}C_3 = \dfrac{19 \times 18 \times 17}{3!} = 969$

Example 81. In how many ways can 20 similar books be placed on 5 different shelves?

[JNTU (K) Oct. 2017, 2019 (Set no. 3)]

Solution: Here $r = 20$ and $n = 5$

$\therefore$ Required number of ways $= C(n + r - 1, r)$

$= C(5 + 20 - 1, 20) = C(24, 20)$

$= \dfrac{24!}{20!4!} = \dfrac{24 \times 23 \times 22 \times 21 \times 20!}{(20!)(24)}$

$= 23 \times 22 \times 21 = 10,626$

Example 82. Find the number of ways of giving 15 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total number of boxes given to A and B does not exceed 6.

[JNTU (K) Oct. 2017 (Set no. 3)]

Solution: Suppose we are given r boxes to A and B together from 15 identical gift boxes. Then, we have $0 \leq r \leq 6$.

Now, the number of ways of giving r boxes to A and B

$= C(n + r - 1, r) = C(2 + r - 1, r) = C(r + 1, r)$ $[\because n = 2]$

∴ The number of ways in which the remaining $(15 - r)$ boxes can be given to C, D, E, F = $C(n + r - 1, r) = C(4 + (15 - r) - 1, r)$ [$\because n = 4$]

$= C(18 - r, r)$

Since $0 \le r \le 6$, the total number of ways in which the gift boxes may be given is $\sum_{r=0}^{6}(r+1) \times C(18 - r, r)$, using Sum Rule.

Example 83. How many solutions does the equation

$$x + y + z = 17$$

have, where x, y, z are non-negative integers?

Solution. The number of integral solution of the given equation is the same as the number of ways of distributing 17 identical things among 3 persons. Hence the required number of solutions

$$= {}^{17+3-1}C_{3-1} = {}^{19}C_2 = \frac{19 \times 18}{2} = 171$$

Example 84. How many solutions are there of $x + y + z = 17$ subject to the constraints $x \ge 1, y \ge 2$ and $z \ge 3$.

Solution. Put $x = 1 + u, y = 2 + v$ and $z = 3 + w$. The given equation becomes $u + v + w = 11$ and we seek in non-negative integers u, v, w. The required number of solution is

$$= {}^{11+3-1}C_{3-1} = {}^{13}C_2 = \frac{13 \times 12}{2} = 78.$$

Example 85. A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

[*JNTU(K) Oct. 2018 (Set No. 4)*]

Solution: The 12 different symbols can be arranged in 12! ways. We know that there are 11 positions between the 12 symbols. Since there must be at least three spaces between successive symbols, 33 out of the 45 blank spaces will be used up. The remaining 12 spaces are to be accommodated in 11 positions. This can be done in $C(n + r - 1, r) = C(11 + 12 - 1, 12) = C(22, 12)$ ways.

Thus, by the product rule, the required number $= 12! \times C(22, 12)$

$$= 12! \times \frac{22!}{12!10!} = \frac{22!}{10!}$$

Mixed Problems on Permutation and Combinations

Example 86. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Solution. Three consonants from 7 consonants can be chosen in ${}^7C_3$ ways

Two vowels from 4 vowels can be chosen in ${}^4C_2$ ways

∴ 3 constants and 2 vowels can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thuse, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels.

Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

$\therefore$ The total number of words = ${}^7C_3 \times {}^4C_2 \times 5! = 25200$.

7.6 NUMBER OF ONTO FUNCTIONS

We know the total number of functions from set A having n elements to a set B of m elements is m^n. If $m \geq n$, then the number of one-one function is $m!/(m-n)!$. The principle of inclusion-exclusion can be used to obtain the number of onto functions from one finite set to another.

Let A and B be two sets having n and $m(\leq n)$ elements respecitively. Then there are

$$m^n - {}^mC_1(m-1)^n + {}^mC_2(m-2)^n + \ldots + (-1)^{m-1}\, {}^mC_{m-1}.$$

onto functions from A to B.

Example 87 In how many ways five different jobs can be assigned to four persons if each person is assigned at least one job?

Solution. Let A be the set of five different jobs and B be the set of four persons. Since each person is to be assigned at least one job, this is equivalent to finding onto functions from the set A to B. Here $|A| = 5$ and $|B| = 4$. Then there are

$4^5 - {}^4C_1(4-1)^5 + {}^4C_2(4-2)^5 - {}^4C_3(4-3)^5$

$= 1024 - 4 \times 243 + 6 \times 32 - 4$

$= 1024 - 972 + 192 - 4$

$= 1216 - 976 = 240$ ways to assign the jobs.

Unit-4 M.F.C.S   II - B.tech   2023-24

Recurrence relation (R.R)

generating functions :-

* The generating function of a sequence. a_0, a_1, a_2, a_3 ----- a_n of a real numbers. is written as. the series. the given below.

$G(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + --- + a_n z^n$

$G(z) = \sum_{n=0}^{\infty} a_n z^n$

find the generating function for the sequence $1, 3, 3^2, 3^3$ ------ (or) find the generating function for the sequence. $\{a_n\}$ with $a_n = 3^n$.

Sol:- given series $1, 3, 3^2, 3^3$ ----

$a_n = 3^n$,

The generating function of given series is $G(z) = \sum_{n=0}^{\infty} 3^n z^n$

Find the generating function for the sequence $1, 2, 3, 4$

Sol:- given series, $1, 2, 3, 4$

$a_n = n+1$.

The generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} (n+1) z^n .$$

find the generating function of the following sequences

(i) 0, 1, -2, 3, -4 - - - -

(ii) 0, 2, 6, 12, 20, 30, 42 - - - -

Sol:- (i) given series, 0, 1, -2, 3, -4, - - - -

$$a_n = (-1)^{n+1} \cdot n$$

The generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \cdot n \, z^n$$

$(-1)^{n+1} \cdot n = 0 \quad n=0$

$(-1)^{1+1} \cdot 1 \; (-1)^{n+1} \cdot n = 1 \quad n=1$

$(-1)^{2+1} \cdot 2 \; (-1)^{n+1} \cdot n = -2$

$(-1)^{3+1} \cdot 3 \Rightarrow n=3$

$(-1)^{4+1} \cdot 4 \Rightarrow n=-4$

(ii) given series; 0, 2, 6, 12, 20, 30, 42

$$a_n = \frac{2n(n+1)}{2}$$

the generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} \frac{2n(n+1)}{2} z^n$$

$n=0 \Rightarrow 0$

$n=1 \Rightarrow \frac{\cancel{2} \cdot 2}{\cancel{2}} = 2$

$n=2 \Rightarrow \frac{\cancel{2}(2)(3)}{\cancel{2}} = 6$

$n=3 \Rightarrow \frac{\cancel{2} \cdot 3 \cdot 4}{\cancel{2}} = 12$

$n=4 \Rightarrow \frac{\cancel{2} \cdot 4 \cdot 5}{\cancel{2}} = 20$

$n=5 \Rightarrow \frac{\cancel{2} \cdot 5 \cdot 6}{\cancel{2}} = 30$

$n=6 \Rightarrow \frac{\cancel{2} \cdot 6 \cdot 7}{\cancel{2}} = 42$

S.no	Sequence (a_n)	generating function $G(z)$
1	a^n	$\frac{1}{1-az}$
2	ka^n	$\frac{k}{1-az}$
3	bna^n	$\frac{baz}{(1-az)^2}$
4	1	$\frac{1}{1-z}$
5	$n+1$	$\frac{1}{(1-z)^2}$
6	$\frac{1}{n!}$	e^z
7	$\frac{(-1)^{n+1}}{n}$	$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$
8	${}^nC_k$	$(1+x)^n$
9	${}^nC_k\, a^n$	$(1+ax)^n$
10	${}^{n-k+1}C_k = {}^{n+k-1}C_{n-1}$	$\frac{1}{(1-x)^n}$
11	$(-)^k\, {}^{n+k-1}C_k = (-1)^k\, {}^{(n+k-1)}C_{n-1}$	$\frac{1}{(1-x)^n}$

problems :-

using generating function to solve the recurrence relation using generating function $a_n = 3a_{n-1} + 2$, $n \geq 1$ with $a_0 = 1$

Sol: given $a_n = 3a_{n-1} + 2$ $n \geq 1$ with $a_0 = 1$

Taking bothsides $\sum_{n=0}^{\infty} z^n$

$$\sum_{n=0}^{\infty} a_n z^n = 3\sum_{n=0}^{\infty} a_{n-1} z^n + 2\sum_{n=0}^{\infty} 1 z^n$$

$$\sum_{n=1}^{\infty} a_n z^n = 3z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} + 2\sum_{n=1}^{\infty} z^n$$

$0 \cdot z^{n}$

a_0

$$(G(z) - a_0) = 3z\,G(z) + 2\frac{1}{(1-z)}$$

$$G(z) - 1 - 3z\,G(z) = \frac{2z}{1-z}$$

$$G(z)\,[1 - 3z] = \frac{2z}{1-z} + 1$$

$$G(z) = \frac{2z + 1 - z}{(1-z)(1-3z)}$$

$$G(z) = \frac{z+1}{(1-z)(1-3z)}$$

$3 \cdot \sum_{n=1}^{\infty} a_{n-1} z^n \frac{z}{z}$

$3z^2$ $\quad \frac{z^n z^{-1}}{z}$

z^{n-1}

$G(z) = a_0 + a_1 z + \cdots$

$G(z) - a_0 = a_1 z + a_2 z^2$

$G(z) - a_0 - a_1 z = a_2 z^2$

$$G(z) = \frac{z+1}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)} \longrightarrow (1)$$

$$\frac{z+1}{\cancel{(1-z)(1-3z)}} = \frac{A(1-3z) + B(1-z)}{\cancel{(1-z)(1-3z)}}$$

$$z + 1 = A(1-3z) + B(1-z) \longrightarrow (2)$$

put $z = 1$ in eq (2) we get

$\cancel{2} = A(\cancel{-2}) + 0$

$$\boxed{A = -1}$$

put $z = \frac{1}{3}$ in eq (2) we get

$$\frac{1}{3} + 1 = 0 + B(1 - \frac{1}{3})$$

$$\cancel{4}^{2}/\cancel{3} = B \frac{\cancel{2}}{\cancel{3}}$$

$$\boxed{B = 2}$$

$$G(z) = \frac{z+1}{(1-z)(1-3z)} = \frac{-1}{1-z} + \frac{2}{1-3z}$$

$$G(z) = \frac{-1}{1-z} + \frac{2}{1-3z}$$

$$G(z) = -1\left(\frac{1}{1-z}\right) + 2\left(\frac{1}{1-3z}\right)$$

$$a_n = -1(1) + 2(3^n)$$

$$\boxed{a_n = -1 + 2(3^n)}$$

② using the method of generating function to solve recurrence relation of

$a_n - 2a_{n-1} - 3a_{n-2} = 0,\ n \geq 2;$ with $a_0 = 3, a_1 = 1$

sol:- given,

$$a_n - 2a_{n-1} - 3a_{n-2} = 0\ n \geq 2,$$

$$\sum_{n=2}^{\infty} a_n z^n - 2\sum_{n=2}^{\infty} a_{n-1} z^n - 3\sum_{n=2}^{\infty} a_{n-2} z^n = 0$$

$$(G(z) - a_0 - a_1 z) - 2z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} - 3z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n}$$

$$(G(z) - a_0 - a_1 z) - 2z\,(G(z) - a_0) - 3z^2 G(z) = 0$$

$$(G(z) - 3 - z) - 2z\,(G(z) - 3) - 3z^2 G(z) = 0$$

$$G(z)\left[-3z^2 - 2z + 1\right] - 3 - z + 6z = 0$$

$$G(z)\left[-3z^2 - 2z + 1\right] - 3 + 5z = 0$$

$$G(z) = \frac{3 - 5z}{(-3z^2 - 2z + 1)}$$

$$G(z) = \frac{3 - 5z}{(1 + z)(1 - 3z)}$$

$$G(z) = \frac{3 - 5z}{(1 + z)(1 - 3z)} = \frac{A}{(1 + z)} + \frac{B}{(1 - 3z)} \longrightarrow (1)$$

$$\frac{3 - 5z}{\cancel{(1 + z)(1 - 3z)}} = \frac{A(1 - 3z) + B(1 + z)}{\cancel{(1 + z)(1 - 3z)}}$$

$$3 - 5z = A(1 - 3z) + B(1 + z) \longrightarrow (2)$$

put $z = -1$ in eq ②, we get

$$3 - 5(-1) = A(1 - 3(-1)) + B(1 - 1)$$

$$3 + 5 = A(1 + 3) + B(0)$$

$$\cancel{8}^{\,2} = A(\cancel{4})$$

$$\boxed{A = 2}$$

Put $z = 1/3$ in eq ②, we get

$$3 - 5(1/3) = A(1 - 3(1/3)) + B(1 + 1/3)$$

$z = -5$ $3 - 5/3 = A(1 - 3/5) + B($

$3 - 5/3 = A(1-1) + B(4/3)$

$\frac{9-5}{3} = B(4/3)$

$4/3 = B(4/3)$

$\boxed{B=1}$

$G(z) = \frac{2}{1+z} + \frac{1}{1-3z}$

$G(z) = 2\left(\frac{1}{1-(-z)}\right) + \frac{1}{1-3z}$

$a_n = 2(-1^n) + (3^n)$

$a_n = -2 + 3^n$

Recurence relation :-

An Equation that Express a_n in terms of one or more of the previous terms of the sequence $a_0, a_1, a_2 --- a_n$ is called a recurrence relation for the sequence $\{a_n\}$.

1) find the first five terms of the sequence define by each of the following recurrence relation and initial conditions

(i) $a_n = a_n^2 - 1$, $a_1 = 2$

(ii) $a_n = n a_{n-1} + n^2 a_{n-2}$ $a_0 = 1$, $a_1 = 1$

(iii) $a_n = a_{n-1} + a_{n-3}$ $a_0 = 1$, $a_1 = 2$, $a_2 = 0$

(i)
sol: given R.R is $a_n = a_{n-1}^2$

put $n = 2$

$a_2 = a_{2-1}^2$

$a_2 = a_1^2$

$a_2 = 4$

$a_3 = a_2^2 = 16$

$a_4 = a_3^2 = (16)^2 = 256.$

$a_5 = a_4^2 = (256)^2 = (65536)$

$a_6 = a_5^2 = (65536)^2 = 4294967296.$

(ii) $a_n = n a_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

given R.R is $a_n = n a_{n-1} + n^2 a_{n-2}$

put $n = 2$

$a_2 = 2 a_{2-1} + 2^2 a_{2-2}$

$a_2 = 2a_1 + 4 a_0$

$a_2 = 2(1) + 4(1)$

$a_2 = 2 + 4$

$a_2 = 6$

$\Rightarrow a_3 = 3 a_{3-1} + 3^2 a_{3-2}$

$= 3 a_2 + 9 a_1$

$= 3(6) + 9(1)$

$= 18 + 9 \Rightarrow 27$

$$a_4 = 4a_{4-1} + (4)^2 a_{4-2}$$
$$= 4a_3 + 16a_2$$
$$= 4(27) + 16(6)$$
$$= 108 + 96$$
$$= 204$$

$$a_5 = 5a_{5-1} + (5)^2 a_{5-2}$$
$$= 5a_4 + 25a_3$$
$$= 5(204) + 25(27)$$
$$= 1020 + 675$$
$$a_5 = 1695$$

$$a_6 = 6(a_5) + 36a_4$$
$$= 6(1695) + 36(204)$$
$$= 17514$$

(iii) given RR is, $a_n = a_{n-1} + a_{n-3}$, $a_0=1, a_1=2, a_2=0$

put $n = 3$

$\rightarrow a_3 = a_{3-1} + a_{3-3}$
$$a_3 = a_2 + a_0$$
$$a_3 = 0 + 1$$
$$a_3 = 1$$

$\rightarrow a_4 = a_{4-1} + a_{4-3}$
$$a_4 = a_3 + a_1$$
$$a_4 = 1 + 2$$
$$a_4 = 3$$

$\rightarrow a_5 = a_{5-1} + a_{5-3}$
$$= a_4 + a_2 \Rightarrow 3 + 0 \Rightarrow 3$$

$\rightarrow a_6 = a_{6-1} + a_{6-3}$

$= a_5 + a_3$

$= 3 + 1$

$a_7 = 4$

$\rightarrow a_7 = a_{7-1} + a_{7-3}$

$= a_6 + a_4$

$= 4 + 3$

$a_7 = 7$

By using an iterative approach find the solutions to each of these recurrence relation with the given initial conditions

(i) $a_n = a_{n-1} + 2$, $a_0 = 3$

(ii) $a_n = a_{n-1} + n$, $a_0 = 1$

(iii) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$

(iv) $a_n = 3a_{n-1} + 1$, $a_0 = 1$

(i) given R.R is $a_n = a_{n-1} + 2$

put $n = 1$

$a_1 = a_0 + 2$

$a_1 = 3 + 2$

$a_1 = 5$

put $n = 2$

$a_2 = a_1 + 2$

$a_2 = 5 + 2$

$= 7$

put $n=3$

$a_3 = a_2+2$

$a_3 = 9$

$3+0\times2$

$3+1\times2$

$3+2\times2$

$3+3\times2$

$3+4\times2$

$3+n\cdot2$

$a_n = 3+2n$

$\rightarrow$ it will satisify from, 0, 1, 2

(ii) given, $a_n = a_{n-1}+n$

given, R.R is $a_n = a_{n-1}+n$

put $n=1$

$a_1 = a_{1-1}+1$

$= a_0+1$

$= 1+1$

$= 2$

put $n=2$

$a_2 = a_{2-1}+2$

$= a_1+2$

$= 2+2$

$= 4$

put $n=3$

$a_3 = a_{3-1}+3$

$a_3 = a_2+3$

$= 4+3$

$a_3 = 7$

put $n=4$

$$a_4 = a_{4-1} + 4$$
$$= a_3 + 4$$
$$= 7 + 4$$
$$= 11$$

$1 + \frac{n+1}{2} \cdot n$

$\Rightarrow$ put $n=1 \Rightarrow 1 + \frac{1+1}{2} \cdot 1 \Rightarrow 2$

$\Rightarrow$ put $n=2 \Rightarrow 1 + \frac{2+1}{2} \cdot 2 \Rightarrow 4$

$\Rightarrow$ put $n=3 \Rightarrow 1 + \frac{3+1}{2} \cdot 3 \Rightarrow 7$

$\Rightarrow$ put $n=4 \Rightarrow 1 + \frac{4+1}{2} \cdot 4 \Rightarrow 11$

$$a_n = 1 + \left(\frac{n+1}{2}\right) \cdot n$$

(iii)

given R.R is $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$

put $n = 1$

$$a_1 = a_{1-1} + 2(1) + 3$$
$$= a_0 + 2 + 3$$
$$= a_0 + 5$$
$$= 4 + 5$$
$$= 9$$

put $n=2$

$$a_2 = a_{2-1} + 2(2) + 3$$
$$= a_1 + 4 + 3$$
$$= a_1 + 7$$
$$= 9 + 7$$
$$a_2 = 16$$

put $n=3$

$$a_3 = a_{3-1} + 2(3) + 3$$
$$= a_2 + 6 + 3$$
$$= a_2 + 9 = 16 + 9$$
$$= 25.$$

put $n=4$

$a_4 = a_{4-1} + 2(4)+3$

$= a_3 + 8 + 3$

$= a_3 + 11$

$= 25 + 11$

$= 36$

$\Rightarrow a_0 = n^2 + 0\times4 + 4 \Rightarrow n=0$

$\Rightarrow a_1 = n^2 + 1\times4 + 4 \Rightarrow n=1$

$\Rightarrow a_2 = n^2 + 2\times4 + 4 \Rightarrow n=2$

$\Rightarrow a_3 = n^2 + 3\times4 + 4 \Rightarrow n=3$

$\vdots$

$a_n = n^2 + n4 + 4$

$a_n = n^2 + 4n + 4$

(iv) given R.R is $a_n = 3a_{n-1} + 1$, $a_0 = 1$

put $n=1$

$a_1 = 3a_{1-1} + 1$

$= 3a_0 + 1$

$= 3(1) + 1$

$= 3 + 1$

$a_1 = 4$

put $n=2$

$a_2 = 3a_{2-1} + 1$

$= 3a_1 + 1$

$= 3(4) + 1$

$= 13$

put $n=3$

$a_3 = 3a_{3-1} + 1$

$a_3 = 3a_2 + 1$
$= 3(13) + 1$
$= 39 + 1$
$= 40$

put $n = 4$
$a_4 = 3a_3 + 1$
$= 3(40) + 1$
$= 121$

$$a_n = \frac{3^{n+1} - 1}{2}$$

put $n=0 \Rightarrow a_n = \frac{3^{n+1}-1}{2}$

$a_n = \frac{3^{0+1}-1}{2}$

$a_n = \frac{3-1}{2} = 1$

put $n=1 \Rightarrow a_n = \frac{3^{1+1}-1}{2}$

$= \frac{3^2-1}{2}$

$= \frac{9-1}{2}$

$= 4$

put $n=2 \Rightarrow a_n = \frac{3^{2+1}-1}{2}$

$= \frac{3^3-1}{2}$

$= \frac{27-1}{2}$

$= 13$

put $n=3 \Rightarrow a_n = \frac{3^{3+1}-1}{2}$

$a_n = \frac{3^4-1}{2}$

$a_n = \frac{80}{2}$

$a_n = 40$

put $n=4, \Rightarrow a_n = \frac{3^{4+1}-1}{2}$

$a_n = \frac{3^5-1}{2}$

$a_n = 121.$

characteristic roots : consider the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $c_1, c_2, c_3 \cdots c_k$ are real numbers,

$\longrightarrow$ The characteristic equation of Recurrence relation, $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots a_c = 0$

$\longrightarrow$ The solutions of characteristic equations are three types

(1) if roots are real & different then the solution is,

$$a_n = c_1 r_1^n + c_2 r_2^n$$

(2) if roots are real & equal then solution is,

$$a_n = (c_1 + c_2 n) r^n$$

(3) if roots are complex roots, then solution is,

$$a_n = r^n [c_1 \cos n\theta + c_2 \sin n\theta]$$

Problems :-

1. solve the Recurrence relation, $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

Sol :- given R.R is,

$$a_n = 5a_{n-1} + 6a_{n-2}$$

By simplifying,

$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$

characteristic eq of R.R is $r^2 - 5r + 6 = 0$

roots, $r = 2, 3$

∴ the given roots are real and different, then the solution is

$$a_n = c_1 r_1^n + c_2 r_2^n$$

$$a_n = c_1(2)^n + c_2(3)^n$$

Now, put $n=0$

$$a_0 = c_1 2^0 + c_2 3^0$$

$$1 = c_1 + c_2 \longrightarrow (1)$$

Now, put $n=1$

$$a_1 = c_1 2^1 + c_2 3^1$$

$$0 = 2c_1 + 3c_2 \longrightarrow (2)$$

Now (1) & (2) becomes,

$$c_1 + c_2 = 1 \times 2$$

$$2c_1 + 3c_2 = 0 \times 1$$

$$2c_1 + 2c_2 = 2$$

$$2c_1 + 3c_2 = 0$$

$$- \quad - \quad -$$

$$-c_2 = 2$$

$$\boxed{c_2 = -2}$$

from, $-2 + c_1 = 1$

$$c_1 = 1 + 2$$

$$\boxed{c_1 = 3}$$

∴ $a_n = 3\, 2^n - 2 \cdot 3^n$.

2) solve the recurrence relation of

$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad n \geq 2 \quad a_0 = 5, a_1 = 12,$

$n > i = 2$

<u>Sol</u> :- Given $a_0 = 5$

$a_1 = 12$

R.R is $a_n - 6a_{n-1} + 9a_{n-2} = 0$

characteristic equation

$$r^2 - 6r + 9 = 0$$

$$r^2 - 3r - 3r + 9 = 0$$

$$r(r-3) - 3(r-3) = 0$$

$$(r-3)(r-3) = 0$$

roots = 3, 3

the given roots are real and equal

the solution will be

$$a_n = (c_1 + c_2 n) 3^n$$

put $n=0$, $a_0 = (c_1 + c_2(0)) 3^0$

$$5 = c_1 \longrightarrow (1)$$

put $n=1$, $a_1 = (c_1 + c_2(1)) 3^1$

$$12 = (c_1 + c_2) 3 \longrightarrow (2)$$

$\cancel{c_1 = 15}$

$5 = c_1$

$12 = (c_1 + c_2) 3$

$$15 + 3c_2 = 12$$

$$3c_2 = 12 - 15$$

$$3c_2 = -3$$

$$c_2 = -1$$

$$a_n = (5-1) 3^n$$

3) solve the recurence relation :

$a_n = 8a_{n-1} - 16a_{n-2}$ for $n \geq 2$, $a_0 = 16$, $a_1 = 80$.

sol :

given R.R is

$a_n = 8a_{n-1} - 16a_{n-2}$

By simplifying,

$a_n - 8a_{n-1} + 16a_{n-2} = 0$

characteristic eq of R.R is, $r^2 - 8r + 16 = 0$

$r^2 - 8r + 16 = 0$

$r^2 - 4r - 4r + 16 = 0$

$r(r-4) - 4(r-4) = 0$

$(r-4)(r-4) = 0$

$r = 4, 4$

The given roots are real and equal the solution will be,

$a_n = (c_1 + c_2 n) 4^n$

put $n=0$

$a_0 = (c_1 + c_2(0)) 4^0$

$16 = c_1$

put $n=1$.

$a_1 = (c_1 + c_2(1)) 4^1$

$80 = (16 + c_2) 4$

$64 + 4c_2 = 80$

$4c_2 = 80 - 64$

$c_2 = 4$

$$a_n = (16+4)\,4^n$$

(4) solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3} \quad \text{for } n = 3,4,5 \ldots$$

with $a_0 = 3,\ a_1 = 6,\ a_2 = 0$

Sol:- given recurrence relation is

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

By simplifying,

$$a_n - 2a_{n-1} + a_{n-2} + 2a_{n-3} = 0$$

characteristics of recurrence relation is

$$r^3 - 2r^2 - r + 2 = 0$$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

roots $= 1, 2, -1$

The roots are real & different $r^2 - r - 2 = 0$

The solution will be

$$r^2 - 2r + r - 2 = 0$$

$$r(r-2) + 1(r-2) = 0$$

$$a_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n \qquad (r-2)(r+1) = 0$$

$$a_n = c_1(1)^n + c_2(2)^n + c_3(-1)^n \qquad r = 2, -1$$

$$a_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n$$

put $n = 0$

$$a_0 = c_1(1)^0 + c_2(-1)^0 + c_3(2)^0$$

$$a_0 = c_1 1^0 + c_2(-1)^0 + c_3(2)^0.$$

$$3 = c_1 + c_2 + c_3 \longrightarrow (1)$$

put $n = 1$

$$a_1 = c_1(1)^1 + c_2(-1)^1 + c_3(2)^1$$

$a_1 =$

$$6 = c_1 - c_2 + 2c_3 \longrightarrow (2)$$

put $n=2$

$$a_2 = c_1(1)^2 + c_2(-1)^2 + c_2(2)^2 \longrightarrow (3)$$

$$0 = c_1 + c_2 + 4c_3 \longrightarrow (3)$$

solve (1) & (2)

$$\begin{aligned} c_1 + \cancel{c_2} + c_3 &= 3 \\ c_1 - \cancel{c_2} + 2c_3 &= 6 \\ \hline 2c_1 + 3c_3 &= 9 \longrightarrow (4) \end{aligned}$$

solve (2) & (3)

$$\begin{aligned} c_1 - \cancel{c_2} + 2c_3 &= 6 \\ c_1 + \cancel{c_2} + 4c_3 &= 0 \\ \hline 2c_1 + 6c_3 &= 6 \longrightarrow (5) \end{aligned}$$

solve (4) & (5)

$$\begin{aligned} 2c_1 + 3c_3 &= 9 \\ 2c_1 + 6c_3 &= 6 \\ \hline -3c_3 &= 3 \\ c_3 &= -1 \end{aligned}$$

sub c_3 value in (5)

$$2c_1 + 6(-1) = 6$$

$$2c_1 = 12$$

$$\boxed{c_1 = 6}$$

sub c_1 & c_3 in (2)

$$6 - c_2 - 2 = 6$$

$$-c_2 = 2$$

$$\boxed{c_3 = -1}$$

$$\boxed{c_2 = -2}$$

solutions of inhomogenous recurrence relation.

→ A linear inhomogenous or non homogenous recurreence reation with constant coefficents of degree k is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + G(n)$, where c_1, c_2 up to c_k are real numbers and equal G(n) is a function not identically zero depending only on (n)

Particular solution for G(n) :-

G(n)	P.I
① constant c	constant d
② linear function$(c_0+c_1 n)$	$d_0+d_1 k$
③ m^{th} degree polynomial $c_0+c_1 n+c_2 n^2+\cdots+c_m n^m$	m^{th} degree polynomial $d_0+d_1 k+d_2 k^2+\cdots+d_m k^m$
④ r^n $r \in R$	$d r^n$

1) solve the recurrence relation

$$a_n = 3a_{n-1} + 2^n, a_0 = 1, n \geq 1$$

sol :- Given $a_n = 3a_{n-1} + 2^n$

it is a non homogenous linear Equation.

$$a_n - 3a_{n-1} = 2^n$$

general solution

$$a_n - 3a_{n-1} = 0$$

The characteristic equation of given eq,

$$r - 3 = 0$$

roots :

$$r = 3$$

The roots are real solution will be,

$$a_n = c_1 (3)^n$$

put $n=0$

$$a_0 = c_1 (3)^0$$

$$1 = c_1$$

$$\boxed{c_1 = 1}$$

$$a_n = (3)^n$$

Now, we can PI,

$$PI = 2^n$$

$$d2^n - 3d\,2^{n-1} = 2^n$$

$$\cancel{2^n}\left(d - \frac{3d}{2}\right) = \cancel{2^n}$$

$$2d - 3d = 2$$

$$-d = 2$$

$$\boxed{d = -2}$$

this is of the form dr^n

$$dr^n = (-2)\,2^n \Rightarrow PI$$

Now, $a_n = G + PI$

$$a_n = (3)^n + (-2)\,2^n$$

Binomial theorem :-

let x & y be any two variables and 'n' be a non-negative integer. Then,

$$\boxed{(x+y)^n = \sum_{i=0}^{n} {}^nC_i \, x^{n-1} y^i}$$

Problem

1) find the coefficent of x^5y^8 in $(x+y)^{13}$.

Given $x^5 y^8$ in $(x+y)^{13}$

By defination of Binomial theorem.

$$(x+y)^n = \sum_{r=0}^{n} nc_r \, x^{n-r} y^r$$

$$(x+y)^n = nc_r \, x^{n-r} y^r$$

$$\Rightarrow n-r=5 \quad , \quad r=8$$

$$13-r=5$$

$$13-8=5$$

$$5=5$$

$$1$$

$$13c_8 \, x^5 y^8 ,$$

$$13c_8 \Rightarrow \frac{13\times12\times11\times10\times9\times1\cancel{8\times9\times8\times7\times6\times5\times4\times3\times2\times1}}{5! \quad \cancel{8\times7\times6\times5\times4\times3\times2\times1}}$$

$$\left\{ \Rightarrow \frac{13\times12\times11\times\cancel{10}\times9\times\cancel{8}\times9}{\cancel{5}\times\cancel{2}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1} \right\} \text{wrong}$$

$$\Rightarrow \frac{13\times\cancel{12}\times11\times\cancel{10}\times9}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1}$$

$$\Rightarrow 1287 \, x^5 y^8$$

$x^{\cdot} \cdot y^{\cdot\cdot} (2x-3y)^{200}$.

Given $x^{101}, y^{99} (2x-3y)^{200}$

By defination of Binomial theorem

$(x+y)^i = \sum_{i=0}^{n} nc_i \, x^{n-i} y^i$

$(x+y)^i = nc_i \, x^{n-i} y^i$

$\Rightarrow 200c_{99} \, (2x)^{101} (-3y)^{99}$

$\Rightarrow 200c_{99} \, 2^{101} \, (+3)^{99} \, x^{101} \, (+y)^{99}$

(3) if n is a non negative integer. show that $\sum_{i=0}^{n} nc_i = 2^n$.

Sol :- By $(x+y)^n = \sum_{i=0}^{n} nc_i \, x^{n-i} y^i$

put $x=1$ & $y=1$

$(2)^n = \sum_{i=0}^{n} nc_i \, (1)^{n-i} (1)^i$

$(2)^n = \sum_{i=0}^{n} nc_i$

(4) if n is a non negative integer show that,

$\sum_{i=0}^{n} (-1)^i nc_i = 0$

Sol :- By $(x+y)^n = \sum_{i=0}^{n} nc_i \, x^{n-i} y^i$

put $x=1$ & $y=-1$

$$0 = \sum_{i=0}^{n} {}^nc_i \, (1)^{n-i} (-1)^i$$

$$0 = \sum_{i=0}^{n} {}^nc_i \, (-1)^i$$

(5) if n is a non-negative integer, show that, $\sum_{i=0}^{n} {}^nc_i \, 2^i = 3^n$.

<u>sol</u> :- By $(x+y)^n = \sum_{i=0}^{n} {}^nc_i \, x^{n-i} y^i$

put $x=1$ $y=2$

$$(1+2)^n = \sum_{i=0}^{n} {}^nc_i \, (1)^{n-i} (2)^i$$

$$(3)^n = \sum_{i=0}^{n} {}^nc_i \, 2^i$$

$$\sum_{i=0}^{n} {}^nc_i (2)^i = 3^n$$

Multinomial coefficents : given non-negative integers, $k_1, k_2, k_3 \ldots k_m$ and $n = k_1 + k_2 + \ldots + k_m$

The multinomial coefficents is $\binom{n}{k_1!\, k_2!\, k_3! \ldots k_m!}$

i.e's $\binom{n}{k_1!\, k_2!\, k_3! \ldots k_m!} = \dfrac{n!}{k_1!\, k_2!\, k_3! \ldots k_m!}$

find the coefficent of $x^3y^3z^2 (2x-5y+5z)^8$

<u>sol</u> :- given, $x^3 y^3 z^2 (2x-5y+5z)^8$

Here, $n = 8$

$k_1 = 3$

$k_2 = 3$

$k_3 = 2$

We know, that, $\dfrac{n!}{k_1!\, k_2!\, k_3! \ldots k_m!}$

$$\Rightarrow \frac{8!}{3!\,3!\,2!} (2x)^3(-5y)^3(5z)^2$$

$$\Rightarrow \frac{8!}{3!\,3!\,2!} 2^3 x^3 (-5)^3 y^3 5^2 z^2$$

$$\Rightarrow \frac{8!}{3!\,3!\,2!} 2^3(-5)^3(5)^2 \cdot x^3y^3z^2$$

$$\Rightarrow \frac{8\times7\times\cancel{6}\times5\times\overset{2}{\cancel{4}}\times\cancel{3\times2\times1}}{\cancel{3\times2\times1}\times\cancel{3\times2}\times1\times\cancel{2}\times1}$$

find the coefficent of $u^2w^3x^4y^2$ in the expansion $(u+v+2w+x+3y+z)^{11}$.

<u>Sol</u> :- Given $u^2w^3x^4y^2 (u+v+2w+x+3y+z)^{11}$

Here, $n = 11$

$k_1 = 2$

$k_2 = 3$

$k_3 = 4$, $k_4 = 2$.

We know that $\frac{n!}{k_1! k_2! k_3! \ldots k_m!}$

$\Rightarrow \frac{11!}{0!\,3!\,4!\,0!} (v)^0 (w)^0 (2w)^3 (x)^4 (3y)^2 (z)^0$

$\Rightarrow \frac{11!}{0!\,3!\,4!\,2!} v^0 \; 2^3 w^3 \; x^4 \; 3^2 y^2$

$\Rightarrow \frac{11!}{0!\,3!\,4!\,2!} (v^2 w^3 x^4 y^2) 2^3 3^2$

Multinomial theorem:

$\rightarrow$ let $a_1, a_2 \ldots a_m \in R$ & $n \in Z$ with $n \geq 1$ Then

$$(a_1 + a_2 + a_3 + \ldots + a_m)^n = \sum_{\substack{0 \leq k_1, k_2 \ldots k_m \leq n \\ k_1 + k_2 + \ldots + k_m = n}} \binom{n}{k_1, k_2 \ldots k_m} a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}$$

$\rightarrow$ Here the sum is indexed overall ordered 'm' integers $k_1, k_2 \ldots k_m \; \forall \; 0 \leq k_1, k_2 \ldots k_m \leq n$ and

$$k_1 + k_2 + k_3 + \ldots k_n = n$$

$\rightarrow$ let us $(a_1 + a_2 + \ldots + a_m)(a_1 + a_2 + \ldots + a_m) \ldots (a_1 + a_2 + \ldots$ n times. Here choosing the value a_{i1} from the summation in the first factor a_{i2} from the sumation of the second factor

$a_{i1} \; a_{i1} \; a_{21} \; a_{31} \ldots a_{im}$

$\rightarrow$ The nominal simplify $a_1^{k_1} a_2^{k_2} \ldots a_m^{k_m}$ since There are n binomial $\binom{n}{k_1, k_2, \ldots k_m}$

base to make those choice the coefficent of $a_1^{k_1}, a_2^{k_2} \cdots a_m^{k_m}$ is $\binom{n}{k_1, k_2 \cdots k_m}$

Application of inclusion and exclusion principle:

→ let x_i be the subset containing the elements that have property P_i. The no. of Elements with all properties, $P_{i_1}, P_{i_2}, \cdots P_{ik}$

→ Then we have $|x_{i_1} \cap x_{i_2} \cap \cdots \cap x_{ik}| = N(P_{i_1}, P_{i_2} \cdots P_{ik})$ if the number of Elements with none of the Properties $P_1, P_2, \cdots P_n$ is denoted

$N = (P_1', P_2' \cdots P_n')$ and the number of Elements in the set is denoted by N

Then $N(P_1', P_2' \cdots P_n') = N - |x_1 \cup x_2 \cup x_3 \cdots x_n|$

by the principle inclusion and exclusion we have

$$N(P_1', P_2' \cdots P_n') = N - \sum_{1 \le i \le n} N(P_i)$$

$$+ \sum_{1 \le i \le J \le n} N(P_i P_J) - \sum_{1 \le i \le J \le k \le n} N(P_i P_J P_k)$$

$$+ \cdots + (-1)^n N(P_1 P_2 \cdots P_n)$$

- find the no. of primes not exceeding 100 and not divisible by 2, 3, 5 or 7
- How many solutions does $x_1 + x_2 + x_3 \le 11$ have where $x_1; x_2\ x_3$ are non negative integers with $x_1 \le 3$, $x_2 \le 4$ and $x_3 \le 6$
- find the number of integer solution of $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where

$x_1 \geq 2,\ x_2 \geq 3,\ x_3 \geq 4,\ x_4 \geq 2,\ x_5 \geq 0.$

④ find the no. of positive integers where n, where $1 \leq n \leq 2000$ & n is not divisible by 2, 3, or 5 but is divisible by ⑦.

1 sol: let, P_1 be the property that an integer is divisible by 2.

let P_2 be the property that an integer is divisible by 3

let P_3 be the property that an integer is divisible by 5

let P_4 be the property that an integer is divisible by 7

The no. of positive integers not exceed 100. that are not divisible by 2, 3, 5, 7 is

$$N(P_1' P_2' P_3' P_4') = N - N(P_1 P_2 P_3 P_4)$$

$$= N - [N(P_1) + N(P_2) + N(P_3) + N(P_4) - N(P_1 P_2) - N(\ldots$$

$$- N(P_1)(P_4) - N(P_2 P_3) - N(P_2, P_4) - N(P_3 P_4 \ldots$$

$$+ N(P_3 P_4 P_1)$$

$$+ N(P_2 P_3 P_4) + N(P_1 P_2 P_3) + N(P_1 P_2 P_4) - N(P_1 P_2 P_3 P_4 \ldots$$

$$N = 99$$

$$\Rightarrow 99 - \left\{\left[\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{7}\right\rfloor - \left\lfloor\frac{100}{2\times 3}\right\rfloor - \left\lfloor\frac{100}{2\times 5}\right\rfloor - \left\lfloor\frac{100}{2\times 7}\right\rfloor\right.\right.$$

$$- \left\lfloor\frac{100}{3\times 5}\right\rfloor - \left\lfloor\frac{100}{2\cdot 7}\right\rfloor - \left\lfloor\frac{100}{5\cdot 7}\right\rfloor + \left\lfloor\frac{100}{2\times 3\times 5}\right\rfloor + \left\lfloor\frac{100}{3\times 5\times 7}\right\rfloor + \left\lfloor\frac{100}{5\times 7\times 2}\right\rfloor$$

$$+ \left\lfloor\frac{100}{2\times 3\times 7}\right\rfloor - \left\lfloor\frac{100}{2\cdot 3\cdot 5\cdot 7}\right\rfloor$$

$\Rightarrow 99 - 78$

$\Rightarrow 21$

Thus The no of integers not exceeding (100) that are divisible by none of 2, 3, 5, 7 is '21'. Hence the no of primes not exceeding 100 is $21 + 4 \Rightarrow 25$

(4)

Sol:- let P_1 be the property that 'n' is divisible by 2

let P_2 be the property that 'n' is divisible by 3

let P_3 be the property that n is divisible by 5

let P_4 be the property that n is divisible by 7

Now the number +ve integer n $1 \le n < 2000$ That are divisible by 2, 3, 5 is.

$$N(P_1'P_2'P_3') = N - (N(P_1) + N(P_2) + N(P_3) - N(P_1P_2) - N(P_2P_3) - N(P_3P_1) + N(P_1P_2P_3))$$

$$= 2000 - \left[\left\lfloor\frac{2000}{2}\right\rfloor + \left\lfloor\frac{2000}{3}\right\rfloor + \left\lfloor\frac{2000}{5}\right\rfloor - \left\lfloor\frac{2000}{2\times3}\right\rfloor - \left\lfloor\frac{2000}{3\times5}\right\rfloor - \left\lfloor\frac{2000}{5\times2}\right\rfloor + \left\lfloor\frac{2000}{2\times3\times5}\right\rfloor\right]$$

$$= 2000 - [1000 + 666 + 400 - 333 - 200 - 133 + 66]$$

$$= 534$$

Hence the number of positive integer $1 \le n \le 2000$ that are not divisible by $2, 3, 5$ but are divisible by '7' Is.

$$534/7 = 76.$$

(8)

Sol:- let P_1 be the property $x_1 > 3$,

P_2 be the property $x_2 > 4$

P_3 be the property $x_3 > 6$.

The Number of solutions satisifying The Equation $x_1 \le 3, x_2 \le 4, x_3 \le 6$ is

$$N(P_1'P_2'P_3') = N - [n(P_1) + n(P_2) + n(P_3) - n(P_1P_2) - n(P_2P_3) - n(P_3P_1) + n(P_1P_2P_3)]$$

Where N is, the total no. of solutions

$\Rightarrow {}^{n+r-1}C_r$

Here $n=3,\ r=11$

$\rightarrow {}^{3+11-1}C_{11}$

${}^{14-1}C_{11}$

${}^{13}C_{11} \Rightarrow \dfrac{13!}{2!\ 11!}$

$$= \frac{13\times \overset{6}{\cancel{12}} \times \cancel{11\times10\times9\times8\times7\times6\times5\times4\times3\times2\times1}}{\cancel{2}!\ \cancel{\times 11\times 10\times 9\times}\cdots\times 1}$$

$N = 78$

$\rightarrow N(P_1)$ is the Number of solutions i.e. is $x_1 \geq 4$

$n=3,\ r=7$

${}^{n+r-1}C_r$

${}^{3+7-1}C_7$

${}^{10-1}C_7$

${}^{9}C_7$.

$$\Rightarrow \frac{9\times8\times\cancel{7\times6\times5\times4\times3\times2\times1}}{(9-7)!\ \cancel{7\times6\times5\times4\times3\times2\times1}}$$

$$\Rightarrow \frac{9\times\overset{4}{\cancel{8}}\cancel{\times7}}{\cancel{2}\times1} \Rightarrow 36$$

$\Rightarrow N(P_2)$ is the Number of solutions is $x_2 \geq 5$

$n=3,\ r=6$

${}^{n+r-1}C_r$

${}^{3+6-1}C_6 \Rightarrow {}^{8}C_6$

$$\frac{\overset{4}{\cancel{8}} \times 7 \times \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}}{\cancel{2}!\ \cancel{6 \times 5 \times 4 \times 3 \times 2 \times 1}} \Rightarrow 28$$

$\Rightarrow N(P_3)$ is the number of solutions is $x_3 \geq 7$

$n = 3, \ r = 4$

$n+r-1C_r$

$3+4-1C_4$

$7-1C_4$

$6C_4$

$$\Rightarrow \frac{\overset{3}{\cancel{6}} \times 5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{2}!\ \times \cancel{4 \times 3 \times 2 \times 1}}$$

$\Rightarrow 15$

$N(P_1P_2)$ The number of solutions is

$x_1 \geq 4, \ x_2 \geq 5$

~~Here~~, $n = 3, \ r = 2$

$n+r-1C_r = 3+2-1C_2 = 5-1C_2 = 4C_2$

$$= \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2 \times 1}}{\cancel{2}!\ \cancel{2 \times 1}} = 6$$

$N(P_2P_3)$ The number of solution is

$x_2 \geq 5 \quad x_3 \geq 7$

Here $n = 3$.

$\therefore$ $x_2 + x_3 = 12$ but total solutions are 11

$\therefore$ no solutions in this case '0'

$n(P_2 \cdot P_3) = 0.$

$N(P_3P_1)$ The number of solution is

$x_3 \geq 7$, $x_1 \geq 4$, $3c_0 = \frac{3!}{3!\,0!} = 3+0-1c_0 = 2c_0$

Here $n=3$, $r=0$

$x_3 + x_1 = 11$ but total solutions are equal

$\therefore$ solutions in this case

$\therefore n(P_3 - P_1) = 0$

$n(P_1P_2P_3)$ The number of solutions is

$x_1 \geq 4$ $x_2 \geq 5$ $x_3 \geq 6$

$N(P_1'P_2'P_3') = 78 - [36 + 28 + 15 - 6 - 1 - 0 + 0]$

$= 78 - 72$

$= 6.$